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Broj: 174/1
Datum: 21.07.2022. god

UNIVERZITET CRNE GORE

SENATU

CENTAR ZA DOKTORSKE STUDIJE

U prilogu akta dostavljam D2 obrazac za Mr Nikolu Konatara sa sjednicie održane 12. 07. 2022. godine godine.

Dekan,
Prof. dr Predrag Miranović

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Broj: 1743
Datum: 22. 07. 2022 god

Na osnovu člana 64 Statuta Univerziteta Crne Gore, a u vezi sa članom 41 stav 1 Pravila doktorskih studija, Vijeća je donijelo

ODLUKU

I

Utvrđuje se da su ispunjeni uslovi iz člana 38 Pravila doktorskih studija za doktoranda Msc Nikolu Konatara

II

Predlaže se Odboru za doktorske studije sastav komisije za ocjenu doktorske disertacije:

1. Prof. dr Oleg Obradović, redovni profesor Prirodno-matematički fakultet Univerziteta Crne Gore u penziji (naučna oblast: optmalno upravljanje, metode regularizacije, parcijalne diferencijalne jednačine);
2. Prof. dr Darko Mitrović, redovni profesor Prirodno-matematičkog fakulteta Univerziteta Crne Gore (naučna oblast: diferencijalne jednačine, parcijalne diferencijalne jednačine);
3. Prof. dr David Kaljaj, redovni profesor Prirodno-matematičkog fakulteta Univerziteta Crne Gore (naučna oblast: kompleksna analiza, geometrijska teorija funkcija);
4. Doc. dr Goran Popivoda, docent Prirodno-matematičkog fakulteta Univerziteta Crne Gore (naučna oblast: teorija vjerovatnoća, slučajni procesi) i
5. Prof. dr Sanja Konjik, redovni professor Prirodno-matematičkog fakulteta Univerziteta u Novom sadu (naučna oblast: diferencijalne jednačine, parcijalne diferencijalne jednačine).

III

Odluka se dostavlja Odboru za doktorske studije Univerziteta Crne Gore.

Univerzitet Crne Gore Prirodno-matematički fakultet

DEKAN

Prof. dr Predrag Miranović

ISPUNJENOST USLOVA DOKTORANDA

OPŠTI PODACI O DOKTORANDU			
Titula, ime, ime roditelja, prezime	MSc Nikola Radoš Konatar		
Fakultet	Prirodno-matematički fakultet		
Studijski program	Matematika		
Broj indeksa	1/16		
NAZIV DOKTORSKE DISERTACIJE			
Na službenom jeziku	Zakoni održanja u okviru stohastičkih i determinističkih modela		
Na engleskom jeziku	Conservation laws in the framework of stochastic and deterministic models		
Naučna oblast	Parcijalne diferencijalne jednačine		
MENTOR/MENTORI			
Prvi mentor	Prof. dr Darko Mitrović	Prirodno-matematički fakultet, Univerzitet Crne Gore, Crna Gora	Diferencijalne jednačine, Parcijalne diferencijalne jednačine
KOMISIJA ZA PREGLED I OCJENU DOKTORSKE DISERTACIJE			
Prof. dr Oleg Obradović		Prirodno-matematički fakultet, Univerzitet Crne Gore, Crna Gora	Optimalno upravljanje, Metode regularizacije, Parcijalne diferencijalne jednačine
Prof. dr Darko Mitrović		Prirodno-matematički fakultet, Univerzitet Crne Gore, Crna Gora	Diferencijalne jednačine, Parcijalne diferencijalne jednačine
Prof. dr David Kalaj		Prirodno-matematički fakultet, Univerzitet Crne Gore, Crna Gora	Kompleksna analiza, Geometrijska teorija funkcija
Doc. dr Goran Popivoda		Prirodno-matematički fakultet, Univerzitet Crne Gore, Crna Gora	Teorija vjerovatnoća, Slučajni procesi
Prof. dr Sanja Konjik		Prirodno-matematički fakultet, Univerzitet u Novom Sadu, Srbija	Diferencijalne jednačine, Parcijalne diferencijalne jednačine
Datum značajni za ocjenu doktorske disertacije			

Sjednica Senata na kojoj je data saglasnost na ocjenu teme i kandidata	11.12.2020.
Dostavljanja doktorske disertacije organizacionoj jedinici i saglasanost mentora	11.07.2022.
Sjednica Vijeća organizacione jedinice na kojoj je dat prijedlog za imenovanje komisija za pregled i ocjenu doktorske disertacije	12.07.2022.

ISPUNJENOST USLOVA DOKTORANDA

U skladu sa članom 38 pravila doktorskih studija kandidat je cjelokupna ili dio sopstvenih istraživanja vezanih za doktorsku disertaciju publikovao u časopisu sa **(SCI/SCIE)/(SSCI/A&HCI)** liste kao prvi autor.

Konatar N., *UNIQUENESS FOR STOCHASTIC SCALAR CONSERVATION LAWS ON RIEMANNIAN MANIFOLDS REVISITED*, Filomat 36:5 (2022), 1615–1634, https://doi.org/10.2298/FIL2205615K, https://www.pmf.ni.ac.rs/filomat-content/2022/36-5/36-5-15-16265.pdf

Spisak radova doktoranda iz oblasti doktorskih studija koje je publikovao u časopisima sa (upisati odgovarajuću listu)

(dati spisak radova koji sadrži: prezimena i imena autora, naziv naučnog rada, ime izdavača, mjesto i godinu izdavanja, DOI, link ka radu i dokaz za JRC)

1. **Konatar N.,** *UNIQUENESS FOR STOCHASTIC SCALAR CONSERVATION LAWS ON RIEMANNIAN MANIFOLDS REVISITED*, Filomat 36:5 (2022), 1615–1634, https://doi.org/10.2298/FIL2205615K, https://www.pmf.ni.ac.rs/filomat-content/2022/36-5/36-5-15-16265.pdf
2. **Konatar N.,** *SCALAR CONSERVATION LAWS WITH CHARATHEODORY FLUX REVISITED,* Glasnik Matematicki, Vol. 55, No. 1 (2020), 101-111., https://doi.org/10.3336/gm.55.1.09, https://web.math.pmf.unizg.hr/glasnik/vol_55/no1_09.html
3. **Konatar N.,** *Dynamics of three dimensional flow in porous media,* Electron. J. Differential Equations, Vol. 2017 (2017), No. 191, pp. 1-5., https://ejde.math.txstate.edu/Volumes/2017/191/konatar.pdf

Obrazloženje mentora o korišćenju doktorske disertacije u publikovanim radovima

Doktorand Nikola Konatar je, kao prvi autor, dio rezultata sopstvenih istraživanja objavio u radovima koji su publikovani u časopisima indeksiranim na SCI/SCIE listi.

U radu *UNIQUENESS FOR STOCHASTIC SCALAR CONSERVATION LAWS ON RIEMANNIAN MANIFOLDS REVISITED,* objavljenom u časopisu Filomat, kandidat daje novi, jednostavniji dokaz jedinstvenosti dopustivog (kinetičkog) rješenja Košijevog zadatka za stohastički zakon održanja

$$du + div_g f(x,u)dt = \Phi(x,u)dW_t, x\epsilon M, t \geq 0,$$

sa početnim uslovom u_0 na glatkoj, kompaktnoj Rimanovoj mnogostrukosti (M,g), gdje je W_t Vinerov proces, a $x \to f(x,\xi)$ vektorsko polje na M za svako ξ iz skupa realnih brojeva, a Φ funkcija iz prostora $C_0^1(M,R)$.

S obzirom da je kandidat ispunio sve uslove propisane Statutom Univerziteta Crne Gore i Pravilma o doktorskih studija, mentor je saglasan da se imenuje Komisija za pregled u ocjenu doktorske disertacije.

Datum i ovjera (pečat i potpis odgovorne osobe)	
U Podgorici, 22. 7. 2022. g. MP	za DEKANa

Prilog dokumenta sadrži:

1. Potvrdu o predaji doktorske disertacije organizacionoj jedinici
2. Odluku o imenovanju komisije za pregled i ocjenu doktorske disertacije
3. Kopiju rada publikovanog u časopisu sa odgovarajuće liste
4. Biografiju i bibliografiju kandidata
5. Biografiju i bibliografiju članova komisije za pregled i ocjenu doktorske disertacije sa potvrdom o izboru u odgovarajuće akademsko zvanje i potvrdom da barem jedan član komisije nije u radnom odnosu na Univerzitetu Crne Gore

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Datum: 11-07-2022

Na osnovu člana 33 Zakona o upravnom postupku, nakon uvida u službenu evidenciju, Prirodno-matematički fakultet izdaje

POTVRDU

MSc Nikola Konatar, student doktorskih studija na Prirodno-matematičkom fakultetu u Podgorici, dana 11.07.2022. godine, dostavio je ovom fakultetu doktorsku disetaciju pod nazivom „**Zakoni održanja u okviru stohastičkih i determinističkih modela**" na dalje postupanje.

Univerzitet Crne Gore Prirodno-matematički fakultet

Dekan,

Prof. dr Predrag Miranović

UNIVERZITET CRNE GORE

PRIRODNO-MATEMATIČKI FAKULTET

Na osnovu člana 37 Pravila doktorskih studija Univerziteta Crne Gore dajem sljedeću

SAGLASNOST

Doktorska disertacija pod naslovom „*Zakoni održanja u okviru stohastičkih i determinističkih modela*" kandidata MSc Nikole Konatara zadovoljava kriterijume propisane Statutom Univerziteta Crne Gore i Pravilima doktorskih studija.

Mentor

Prof. dr Darko Mitrović

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Uniqueness for Stochastic Scalar Conservation Laws on Riemannian Manifolds Revisited

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Abstract. We revise a uniqueness question for the scalar conservation law with stochastic forcing

$$du + \mathrm{div}_g \mathfrak{f}(x, u)dt = \Phi(x, u)dW_t, \quad x \in M, \quad t \geq 0$$

on a smooth compact Riemannian manifold (M, g) where W_t is the Wiener process and $x \mapsto \mathfrak{f}(x, \xi)$ is a vector field on M for each $\xi \in \mathbb{R}$. We introduce admissibility conditions, derive the kinetic formulation and use it to prove uniqueness in a more straight-forward way than in the existing literature.

1. Introduction

The aim of the paper is to offer a simpler proof of uniqueness of admissible (i.e. kinetic) solution to the Cauchy problem for a stochastic scalar conservation law of the form

$$du + \mathrm{div}_g \mathfrak{f}(x, u)dt = \Phi(x, u)dW_t, \quad x \in M, \quad t \geq 0 \tag{1}$$

$$u|_{t=0} = u_0(x) \in L^\infty(M) \tag{2}$$

on a smooth, compact, d-dimensional (Hausdorff) Riemannian manifold (M, g). The object W is the Wiener process which can be finite or infinite dimensional which does not affect the essence of the proofs.

The proof of well-posedness has been recently presented in [14]. The authors considered the kinetic formulation of (1) and prove the uniqueness by finding a relation between the kinetic function and square of the kinetic function (see [14, (4.13)]). The procedure appeared to be quite complicated and we show here that it is possible to obtain the proof by considering the product of kinetic solution h and the function $(1-h)$.

More precisely, our idea of proof has the same starting point as in [14] since it is based on the appropriate kinetic reformulation of the problem (see (35) below). In [14], the authors then prove that the kinetic function h given by Definition 3.4 satisfies $h^2 = h$. However, unlike the method from [14] where the authors derive the equation for h^2 and then compare it with the equation for h in order to draw conclusions, we obtain

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an equation for $h(1-h)$ and use it to prove the uniqueness. Although the latter sounds the same, the regularization procedures in our situations are easier to follow (as we essentially closely follow the steps from the Euclidean case) and thus the method seems simpler than the one proposed in [14].

The basic reason for the simplification lies in the fact that the equations for h and $(1-h)$ are symmetric which is why we can fairly easily eliminate the terms appearing on the right hand-side of the latter equations and thus reach the Kato inequality (see (45)). Moreover, by regularizing the equation via the convolution with respect to x and ξ we obtain a function which is by assumption continuous with respect to time and we can directly use the Itô formula instead of using its generalized variant (see [14]).

We note that one of the ingredients of the proof is the classical method of the doubling of variables (see [22]). The method could be avoided since we regularized the equation (which means that we can use basic calculus for smooth functions). However, the analysis would then require various adaptations of the Friedrichs lemma (specially in a viewpoint that we have terms with the Wiener measure) and it appears that the proofs would not be easier.

Let us now introduce precise assumptions on the coefficients of the equation. First, we shall assume that we work with one-dimensional Wiener process defined on the stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$. We will also assume that

- the flux $\mathfrak{f} \in C^1(M \times \mathbb{R}; \mathbb{R}^d)$ satisfies the geometry compatibility conditions and a decay property as follows respectively:

$$\mathrm{div}_g \mathfrak{f}(x,\xi) = 0 \quad \text{for every } \xi \in \mathbb{R} \tag{3}$$

$$\|\mathfrak{f}(\cdot,\lambda)\|_{L^\infty(M)} \leq C(1+|\lambda|); \tag{4}$$

- the function Φ is continuously differentiable and it decays to zero at infinity i.e. $\Phi \in C_0^1(M \times \mathbb{R})$, and

$$\sup_{\lambda\in\mathbb{R}} |\Phi(\cdot,\lambda)\lambda| \in L^1(M). \tag{5}$$

Nowadays, we are witnessing a rapid development of stochastic conservation laws and related equations. The rising interest to this field of research is motivated by concrete applications in biology, porous media, finances (see e.g. randomly chosen [1, 4, 32] and references therein) and, in general, any realistic situation in which we cannot determine parameters precisely (i.e. the coefficients of the equations governing the process).

Moreover, such equations have rich mathematical structure and therefore, they are very interesting and challenging from the mathematical point of view. We have numerous results in different directions beginning with the stochastic conservation laws [5, 6, 12, 13, 18, 19, 34], then velocity averaging results for stochastic transport equations [7, 25], stochastic degenerate parabolic equations [15, 36]. We remark that latter list of references is far from complete. As for the stochastic PDEs on manifolds, we mention [2] where the wave equation was considered.

Now we briefly recall the definition of the divergence on a manifold. We suppose that the map $(x,\xi) \mapsto \mathfrak{f}(x,\xi)$, $M \times \mathbb{R} \to TM$ is C^1 and that, for every $\xi \in \mathbb{R}$, $x \mapsto \mathfrak{f}(x,\xi) \in \mathfrak{X}(M)$ (the space of vector fields on M).

In local coordinates, we write

$$\mathfrak{f}(x,\xi) = (f^1(x,\xi), \dots, f^d(x,\xi)).$$

The divergence operator appearing in the equation is to be formed with respect to the metric, so in local coordinates we have (cf. (10) below):

$$\mathrm{div}_g \mathfrak{f}(x,u) = \mathrm{div}_g\left(x \mapsto \mathfrak{f}(x,u(t,x))\right) = \frac{\partial}{\partial x_k}(f^k(x,u(t,x)) + \Gamma^j_{kj}(x) f^k(x,u(t,x)) \tag{6}$$

where the Γ-terms are the Christoffel symbols of g and the Einstein summation convention is in effect.

As we can see, the divergence operator on manifolds is more involved than the one in Euclidean setting. Therefore, in order to prove uniqueness, we need to assume (3). Remark that (3) is the incompressibility condition from the fluid dynamics point of view, because, due to conservation of mass of an incompressible fluid, the density in a control volume changes according to the stochastic forcing

$$\frac{D\rho}{Dt} = \Phi(x,\rho)\frac{dW_t}{dt} \tag{7}$$

where ρ is density of the control volume and $\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \frac{dx}{dt}\cdot\nabla\rho$ is the material derivative for the flow velocity $\frac{dx}{dt} = (\frac{dx_1}{dt},\dots,\frac{dx_d}{dt})$. If we assume that the function ρ is smooth, we can rewrite equation (1) in the form

$$\frac{\partial\rho}{\partial t} + \partial_\xi\big(\mathfrak{f}(x,\xi)\big)\big|_{\xi=\rho}\cdot\nabla_g\rho + \mathrm{div}_g\,\mathfrak{f}(x,\xi)\big|_{\xi=\rho} = \Phi(x,\rho)\frac{dW}{dt}. \tag{8}$$

Then, taking as usual $\frac{dx}{dt} = \partial_\xi\big(\mathfrak{f}(x,\xi)\big)\big|_{\xi=\rho}$ and comparing (8) and (7), we arrive at

$$\mathrm{div}_g\,\mathfrak{f}(x,\xi)\big|_{\xi=\rho} = 0,$$

which immediately gives what is called the geometry compatibility condition.

Since the equation we consider is a nonlinear hyperbolic equation, its solution in general contains discontinuities and we need to pass to the weak solution concept. However, this induces uniqueness issues as one can in general construct several weak solutions satisfying the same initial data. Thus, in order to isolate the physically admissible one, we need to introduce entropy type admissibility conditions [22]. We will first derive them locally and then, using the geometry compatibility conditions, we shall show that the conditions hold globally as well.

Having the admissibility conditions, we can derive the kinetic formulation to (1) (see (33)). We will use it to prove the uniqueness to the considered Cauchy problem. The strategy of proof is adapted from [6]. We have tried to be as precise, self contained and intuitive as possible. We have therefore proven a simple corollary of the Itô lemma concerning the derivative of the product of two stochastic processes and derive the uniqueness proof first informally, and then also formally.

The paper is organized as follows. In Section 2 we introduce notions and notations from differential geometry and stochastic calculus. We then move on to derive the kinetic formulation of (1) and heuristically show how to get uniqueness to the solution. In Section 5, we formally prove the uniqueness result.

2. Preliminaries from Riemannian geometry and stochastic calculus

We shall split the section into two parts. In the first one, we will provide details from differential geometry, and in the second one, we recall necessary results from stochastic calculus.

2.1. *Riemannian geometry*

Our standard references for notions from Riemannian and distributional geometry are [17, 26, 27, 29]. As before, (M, g) will be a d-dimensional Riemannian manifold. If v is a distributional vector field on M then its gradient ∇v is the vector field metrically equivalent to the exterior derivative dv of v: $\langle\nabla v, X\rangle = dv(X) = X(v)$ for any $X \in \mathfrak{X}(M)$. In local coordinates,

$$\nabla v = g^{ij}\frac{\partial v}{\partial x^i}\partial_j, \tag{9}$$

with g^{ij} the inverse matrix to $g_{ij} = \langle\partial_{x^i}, \partial_{x^j}\rangle$.

As for the Laplace-Beltrami operator Δ_g on M, for a function $f \in C^2(M)$ in terms of local coordinates we have

$$\Delta_g f = \nabla_g^2 f = \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j f\right)$$

Finally, the divergence operator on M is locally defined via the Christofel symbols for a C^1 vector field on $X \in \mathcal{T}_0^1 = \mathfrak{X}(M)$ with local representation $X = X^i \frac{\partial}{\partial x^i}$:

$$\operatorname{div} X = \frac{\partial X^k}{\partial x^k} + \Gamma^j_{kj} X^k. \tag{10}$$

To proceed, we shall need basic notions from the Sobolev spaces on manifolds.

Since M is a compact manifold, we can define for a fixed $k \in \mathbb{N}$ (keeping in mind the Poincare inequality)

$$f \in H^k(M) \quad \Leftrightarrow \quad \|\nabla_g^k f\|_{L^2(M)} < \infty.$$

As for for the Sobolev spaces with negative indexes, we have

$$f \in H^{-k}(M) \text{ if } \exists F \in H^k(M) \text{ such that } \Delta^{2k} F = f$$

and we define

$$\|f\|_{H^{-k}(M)} = \|F\|_{H^k(M)}. \tag{11}$$

The spaces $H^k(M)$, $k \in \mathbb{Z}$, are Hilbert spaces and we denote by $\{e_k\}_{k\in\mathbb{N}}$ the orthogonal basis in $L^2(M)$ which is given as the set of eigenfunctions corresponding to the Laplace-Beltrami operator:

$$\Delta_g e_k(x) = -\lambda_k e_k(x).$$

At the same time, the set $\{e_k\}_{k\in\mathbb{N}}$ is the basis in $H^s(M)$, $s \in \mathbb{Z}$, according to the density arguments. We remark that it is usual to take the eigenvectors of the operator $(1 - \Delta_g)$ but since we are on the compact manifold, we can safely work with the simplified version.

Notice that if we have a function $g \in H^k(M)$ and we rewrite it in the basis $\{e_k/\|e_k\|_{H^k(M)}\}$:

$$g(x) = \sum_{k=1}^{\infty} g_k e_k(x)/\|e_k\|_{H^k(M)} \tag{12}$$

then

$$g_k = \int_M g(x) \frac{e_k(x)}{\|e_k\|_{H^k(M)}} dx \tag{13}$$

which is easily obtained by multiplying (12) by $e_k/\|e_k\|_{H^k(M)}$, integrating the result over M and using the orthogonality of $\{e_k/\|e_k\|_{H^k(M)}\}$. Moreover,

$$\|g\|_{H^k(M)} = \sum_{k=1}^{\infty} g_k^2. \tag{14}$$

It is not difficult to notice that according to the definition of e_k and (11), we have

$$\|e_k\|_{L^2(M)} = \sqrt{\lambda_k}\|e_k\|_{H^{-1}(M)}. \tag{15}$$

Let us now recall basic notions from stochastic calculus.

2.2. Stochastic calculus

We start with the notion of predictability for the Hilbert-space valued stochastic processes.

Definition 2.1. *Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space and $\{\mathcal{F}_t\}_{t\in[0,T]}$, $T > 0$, be a filtration of the sigma algebra $\mathcal{F}$. Let V be a fixed Hilbert space with dual V^*.*

We say that the stochastic process $X : \Omega \times [0,T] \to V$ is adapted with respect to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ if for every $\varphi \in V^*$ the stochastic process $\langle X(t), \varphi\rangle$ is measurable with respect to the σ-algebra $\mathcal{F}_t$ for any $t > 0$.

We note that in the latter definition we require the weak measurability of the mapping $X : \Omega \times [0,T] \to V$, but as we are going to deal with the Sobolev spaces $H^k(M)$, $k \in \mathbb{N} \cup \{0\}$ which are separable, the notions of weak and strong measurability coincide (see e.g. [23]). To this end, we use the following notations for $H^1(M)$ and $L^2(M)$-valued square integrable stochastic processes:

$$L^2_{\mathrm{P}}(\Omega; L^2((0,T); H^1(M))) = \{u : (0,T)\times M\times\Omega \to \mathbb{R} : \int_\Omega \int_0^T \|u(t,\cdot,\omega)\|^2_{H^1(M)} dt d\mathrm{P}(\omega) < \infty\}$$

$$L^2_{\mathrm{P}}(\Omega; L^2((0,T)\times M)) = \{u : (0,T)\times M\times\Omega \to \mathbb{R} : \int_\Omega \int_0^T \|u(t,x,\omega)\|^2_{L^2(M)} dt d\mathrm{P}(\omega) < \infty\}$$

In both cases, the required measurability assumptions are tacitly assumed.

Let us now introduce the Itô lemma and some of its corollaries. To this end, let X_t be a stochastic process satisfying the following stochastic differential equation:

$$dX_t = \mu_1 dt + \sigma_1 dW_t. \tag{16}$$

We remark here that the latter equation is actually an informal way of expressing the integral equality

$$X_{t_0+s} - X_{t_0} = \int_{t_0}^{t_0+s} \mu_1 dt + \int_{t_0}^{t_0+s} \sigma_1 dW_t, \quad \forall t_0, s > 0. \tag{17}$$

By Itô's lemma, for each twice differentiable scalar function $f = f(t,z)$ the equation

$$df(X_t) = \left(\frac{\partial f}{\partial t} + \mu_1 \frac{\partial f}{\partial z} + \frac{\sigma_1^2}{2}\frac{\partial^2 f}{\partial z^2}\right) dt + \sigma_1 \frac{\partial f}{\partial z} dW_t \tag{18}$$

holds.

By taking $f(t, X_t) = X_t^2$, we get

$$dX_t^2 = 2\mu_1 X_t dt + \sigma_1^2 dt + 2\sigma_1 X_t dW_t. \tag{19}$$

Notice that $2\mu_1 X_t dt + 2\sigma_1 X_t dW_t = 2X_t(\mu_1 dt + \sigma_1 dW_t) = 2X_t dX_t$, so (19) becomes

$$dX_t^2 = 2X_t dX_t + \sigma_1^2 dt. \tag{20}$$

Similarly, if Y_t is a stochastic process satisfying the stochastic differential equation

$$dY_t = \mu_2 dt + \sigma_2 dW_t \tag{21}$$

then

$$dY_t^2 = 2Y_t dY_t + \sigma_2^2 dt, \tag{22}$$

$$d(X_t + Y_t)^2 = 2(X_t + Y_t) d(X_t + Y_t) + (\sigma_1 + \sigma_2)^2 dt. \tag{23}$$

The left-hand side of (23) is

$$\begin{aligned} d(X_t + Y_t)^2 &= d(X_t^2 + 2X_tY_t + Y_t^2) = dX_t^2 + 2d(X_tY_t) + dY_t^2 \\ &= 2X_t dX_t + \sigma_1^2 dt + 2d(X_tY_t) + 2Y_t dY_t + \sigma_2^2 dt, \end{aligned} \tag{24}$$

and the right side is

$$2(X_t+Y_t)d(X_t+Y_t)+(\sigma_1+\sigma_2)^2dt \\ = 2X_tdX_t+2X_tdY_t+2Y_tdX_t+2Y_tdY_t+\sigma_1^2dt+2\sigma_1\sigma_2dt+\sigma_2^2dt. \quad (25)$$

By annuling the same terms on the left and right side respectively, and dividing the equation by 2, we get

$$d(X_tY_t)=X_tdY_t+Y_tdX_t+\sigma_1\sigma_2dt. \quad (26)$$

Let us finally recall the Itô isometry. The following equality holds

$$E\left[\left(\int_0^T X_t dW_t\right)^2\right]=E\left[\int_0^T X_t^2 dt\right].$$

3. Entropy admissibility and kinetic formulation

Let us first informally derive the admissibility conditions. As usual, we start with the parabolic approximation to (1)

$$du_\varepsilon+\mathrm{div}_g(\mathfrak{f}(\mathrm{x},u_\varepsilon))dt=\Phi(\mathrm{x},u_\varepsilon)dW_t+\varepsilon\Delta_g u_\varepsilon dt,\ \mathrm{x}\in M,\ t\in(0,T) \quad (27)$$

where, as before, $\mathfrak{f}=\mathfrak{f}(x,\lambda)\in C^1(M\times\mathbb{R})$ and (M,g) is a d-dimensional Rimannian manifold with the metric g. We will assume that W_t is a Wiener process and $\Phi\in C_0^1(M\times\mathbb{R})$.

Let us recall the definition of the weak solution to (27), (2).

Definition 3.1. *We say that the measurable function $\Omega\ni\omega\mapsto u_\varepsilon(\cdot,\omega)\in L^2([0,T];H^1(M))$ adapted with respect to the filtration $\{\mathcal{F}_t\}$ is the weak solution to (27), (2) if for a test function $\varphi\in C_0^2([0,T]\times M)$ it holds almost surely*

$$\int_0^T\int_M\left(u_\varepsilon\partial_t\varphi+\mathrm{div}_g(\mathfrak{f}(\mathrm{x},u_\varepsilon))\nabla_g\varphi\right)dxdt=\int_0^T\int_M\varphi\Phi(\mathrm{x},u_\varepsilon)dW_t-\varepsilon\int_0^T\int_M u_\varepsilon\Delta_g\varphi dxdt.$$

Existence of the solution to (27), (2) can be concluded from the general arguments given in [23]. One can also find a proof in [16].

Using the Itô formula, from (27) we get (here and in the sequel, we will set $\mathfrak{f}'(\mathrm{x},\xi)=\partial_\xi\mathfrak{f}(\mathrm{x},\xi)$):

$$d\theta(u_\varepsilon)=\Big(-\theta'(u_\varepsilon)\mathfrak{f}'(\mathrm{x},u_\varepsilon)\cdot\nabla_g u_\varepsilon+\theta'(u_\varepsilon)\,\mathrm{div}_g\,\mathfrak{f}(\mathrm{x},\rho)\big|_{\rho=u_\varepsilon} \\ +\varepsilon\Delta_g\theta(u_\varepsilon)-\varepsilon\theta''(u_\varepsilon)|\nabla_g u_\varepsilon|^2+\frac{\Phi^2(\mathrm{x},u_\varepsilon)}{2}\theta''(u_\varepsilon)\Big)dt+\Phi(\mathrm{x},u_\varepsilon)\theta'(u_\varepsilon)dW_t \quad (28)$$

for all twice differentiable scalar functions θ.

Using the standard approximation procedure and taking into account convexity of the function $\theta(u)=|u-\xi|_+=\begin{cases}u-\xi, & u\geq\xi\\ 0, & else\end{cases}$, we know that we can safely plug it into (28). After letting $\varepsilon\to0$ and assuming that $E(|u_\varepsilon(t,\mathrm{x})-u(t,\mathrm{x})|)\to0$ as $\varepsilon\to0$, we get the following distributional inequality:

$$d|u-\xi|_+\leq-\mathfrak{f}'(\mathrm{x},u)\nabla_g u\,\mathrm{sign}_+(u-\xi)dt+\theta'(u)\,\mathrm{div}_g\,\mathfrak{f}(\mathrm{x},\rho)\big|_{\rho=u}dt \\ +\frac{\Phi^2(\mathrm{x},u)}{2}\delta(u-\xi)dt+\Phi(\mathrm{x},u)\,\mathrm{sign}_+(u-\xi)dW_t. \quad (29)$$

Taking into account the geometry compatibility condition (3), we have

$$\mathfrak{f}'(\mathrm{x},u)\cdot(\nabla_g u)\,\mathrm{sign}_+(u-\xi)=\mathrm{div}_g\left(\mathrm{sign}_+(u-\xi)(\mathfrak{f}(\mathrm{x},u)-\mathfrak{f}(\mathrm{x},\xi))\right)$$
$$+\,\mathrm{sign}_+(u-\xi)\,\mathrm{div}_g\,\mathfrak{f}(\mathrm{x},\xi)=\mathrm{div}_g\left(\mathrm{sign}_+(u-\xi)(\mathfrak{f}(\mathrm{x},u)-\mathfrak{f}(\mathrm{x},\xi))\right), \tag{30}$$

and using the Schwartz lemma on non-negative distributions, we conclude that there exists a non-negative stochastic kinetic measure m (to be precised later) such that the equation (29) can be written as

$$d|u-\xi|_+=-\,\mathrm{div}_g(\mathrm{sign}_+(u-\xi)(\mathfrak{f}(\mathrm{x},u)-\mathfrak{f}(\mathrm{x},\xi)))dt+\frac{\Phi^2(\mathrm{x},u)}{2}\delta(u-\xi)dt$$
$$+\Phi(\mathrm{x},u)\,\mathrm{sign}_+(u-\xi)dW_t-dm(t,\mathrm{x},\xi)dt. \tag{31}$$

Next, we find the partial derivative of the expression given in (31) with respect to ξ to get

$$d\partial_\xi|u-\xi|_+=-\,\mathrm{div}_g(-\mathfrak{f}'(\mathrm{x},\xi)\,\mathrm{sign}_+(u-\xi))dt+\partial_\xi\left(\frac{\Phi^2(\mathrm{x},u)}{2}\delta(u-\xi)\right)dt$$
$$+\partial_\xi(\Phi(\mathrm{x},u)\,\mathrm{sign}_+(u-\xi)dW_t)-\partial_\xi dm. \tag{32}$$

Introducing $h(t,x,\xi)=-\partial_\xi|u-\xi|_+=\mathrm{sign}_+(u-\xi)$ into (32) gives

$$dh+\mathrm{div}_g(\mathfrak{f}'(\mathrm{x},\xi)h)dt=-\partial_\xi\left(\frac{\Phi^2(\mathrm{x},u)}{2}\delta(u-\xi)\right)dt-\partial_\xi(\Phi(\mathrm{x},u)hdW_t)+\partial_\xi dm. \tag{33}$$

Notice that

$$\partial_\xi(\Phi(\mathrm{x},u)hdW_t)=\partial_\xi(\Phi(\mathrm{x},u)\,\mathrm{sign}_+(u-\xi))dW_t=-\Phi(\mathrm{x},u)\delta(u-\xi)dW_t$$
$$=-\Phi(\mathrm{x},\xi)\delta(u-\xi)dW_t. \tag{34}$$

Using $\frac{\Phi^2(\mathrm{x},u)}{2}\delta(u-\xi)=\frac{\Phi^2(\mathrm{x},\xi)}{2}\delta(u-\xi)$ and (34), and denoting the measure $-\partial_\xi h=\delta(u-\xi)$ by $\nu_{(t,\mathrm{x})}(\xi)$, we finally get the weak form of our equation:

$$dh+\mathrm{div}_g(\mathfrak{f}'(\mathrm{x},\xi)h)dt=-\partial_\xi\left(\frac{\Phi^2(\mathrm{x},\xi)}{2}\nu_{(t,\mathrm{x})}(\xi)\right)dt+\Phi(\mathrm{x},\xi)\nu_{(t,\mathrm{x})}(\xi)W_t+\partial_\xi dm\,. \tag{35}$$

We shall call the latter equation *the kinetic formulation* of (1).

It is important to notice that the function $\overline{h}=1-h$ satisfies

$$d\overline{h}+\mathrm{div}_g(\mathfrak{f}'(\mathrm{x},\xi)\overline{h})dt=\partial_\xi\left(\frac{\Phi^2(\mathrm{x},\xi)}{2}\nu_{(t,\mathrm{x})}(\xi)\right)dt-\Phi(\mathrm{x},\xi)\nu_{(t,\mathrm{x})}(\xi)dW_t-\partial_\xi dm. \tag{36}$$

We can now introduce a definition of an admissible solution. Let us first introduce what we meant under the stochastic measure here.

Definition 3.2. *We say that a mapping m from Ω into the space of Radon measures on $[0,T]\times M\times\mathbb{R}$ is a stochastic kinetic measure if:*

- *for every $\phi\in C_0([0,T]\times M\times\mathbb{R})$ the action $\langle m,\phi\rangle$ defines a $\mathbf{P}$-measurable function*

$$\langle m,\phi\rangle:\Omega\to R;$$

- *m vanishes for large ξ: if $B_R^c = \{\xi \in \mathbb{R} \mid |\xi| \geq R\}$, then*

$$\lim_{R\to\infty} Em(C_0([0,T] \times M \times B_R^c)) = 0$$

- *for every $\phi \in C_0(M \times \mathbb{R})$, the process*

$$t \mapsto \int_{[0,t]\times M\times\mathbb{R}} \phi(x,\xi)dm(s,x,\xi)$$

is predictable.

Definition 3.3. *The measurable function $u : [0,T] \times M \times \Omega \to \mathbb{R}$ almost surely continuous with respect to time in the sense that $u(\cdot,\cdot,\omega) \in C(\mathbb{R}^+; H^{-1}(M))$ for P-almost every $\omega \in \Omega$, adapted with respect to the filtration $\{\mathcal{F}_t\}$, is an admissible stochastic solution to* (1), (2) *if*

- *there exists $C_2 > 0$ such that $E(\text{esssup}_{t\in[0,T]} \|u(t)\|_{L^2(M)}) \leq C_2$;*
- *the kinetic function $h = \text{sign}_+(u - \xi)$ adapted with respect to the filtration $\{\mathcal{F}_t\}$ satisfies* (31) *with the initial conditions $h(0,x,\xi) = \text{sign}_+(u_0(x) - \xi)$ in the sense of weak traces and $\bar{h}$ satisfies* (36) *with the initial conditions $\bar{h}(0,x,\xi) = 1 - \text{sign}_+(u_0(x) - \xi)$ in the sense of weak traces.*

We shall also need a notion of the generalized stochastic kinetic solution.

Definition 3.4. *A measurable function $\omega \mapsto h(\cdot,\cdot,\cdot,\omega) \in L^2([0,T] \times M \times K)) \cap C_{LR}([0,T]; H^{-k}(M \times K))$ (with $C_{LR}(X)$ we denote the set of left and right continuous functions on X), for some $k \in \mathbb{N}$ and any $K \subset\subset \mathbb{R}$, adapted with respect to the filtration $\{\mathcal{F}_t\}$, bounded between zero and one and non-strictly decreasing with respect to $\xi \in \mathbb{R}$ such that $h = -\partial_\xi \nu_{(t,x)}$ is the generalized stochastic kinetic solution to* (1), (2) *if there exists a non-negative stochastic kinetic measure m such that h satisfies* (35) *and the initial conditions $h(0,x,\xi) = \text{sign}_+(u_0(x) - \xi)$ in the sense of weak traces.*

Clearly, if we have the admissible solution to (1), (2) then we have the generalized stochastic kinetic solution as well. Interestingly, vice versa also holds which follows from the standard uniqueness arguments (see e.g. [6]). The concept of the generalized solution used here is essentially the same as the one from [14] except that we do not require boundedness of the p-moments, $p \in [1,\infty)$, of the measure $\nu_{t,x}$ (see [14, Definition 3.3]). We note that the equation considered here is somewhat simpler than the one in [14] since we do not have cylindrical Wiener process and we require somewhat stricter conditions on the coefficients (compare in particular (4) and (5) here and [14, (2.1), (2.2), (2.3)]). Although insubstantial, the relaxation of the conditions seems sufficient to avoid additional requirements for the generalized stochastic kinetic solution from Definition 3.4.

4. Informal uniqueness proof – doubling of variables

In this section, we shall informally show how to get uniqueness. Formal proof does not essentially differ from the procedure given in this section but one needs to introduce several smoothing procedures which significantly complicates some steps of the proof. Therefore, for readers' convenience, in this section we essentially explain the basic ideas of the proof. We also remark that, in order to simplify the notation, we will denote by dx the measure on the manifold instead of usual $d\gamma(x)$.

Let $h^1(t,x,\xi)$ and $h^2(t,y,\zeta)$ be two different generalized kinetic solutions to (1), (2) (see Definition 3.4). Then

$$dh^1 + \text{div}_g(\mathfrak{f}'(x,\xi)h^1)dt = -\partial_\xi\left(\frac{\Phi^2(x,\xi)}{2}\nu^1\right)dt + \Phi(x,\xi)\nu^1 dW_t + \partial_\xi dm_1, \tag{37}$$

$$d\overline{h^2} + \text{div}_g(\mathfrak{f}'(y,\zeta)\overline{h^2})dt = \partial_\zeta\left(\frac{\Phi^2(y,\zeta)}{2}\nu^2\right)dt - \Phi(y,\zeta)\nu^2 dW_t - \partial_\zeta dm_2. \tag{38}$$

By (26), the following holds:

$$d(h^1\overline{h^2}) = h^1 d\overline{h^2} + \overline{h^2} dh^1 - \Phi(\mathrm{x},\xi)\Phi(\mathrm{y},\zeta)\nu^1 \otimes \nu^2 dt. \tag{39}$$

Multiplying (37) by $\overline{h^2} = \overline{h^2}(t,\mathrm{y},\zeta)$, (38) by $h^1 = h^1(t,\mathrm{x},\xi)$, adding them and using the geometry compatibility conditions (3), yields

$$\begin{aligned}
&\overline{h^2} dh^1 + h^1 d\overline{h^2} + \overline{h^2}\mathfrak{f}'(\mathrm{x},\xi)\cdot\nabla_{g,\mathrm{x}} h^1 dt + h^1 \mathfrak{f}'(\mathrm{y},\zeta)\cdot\nabla_{g,\mathrm{y}}\overline{h^2} dt \\
&= -\overline{h^2}\partial_\xi\left(\frac{\Phi^2(\mathrm{x},\xi)}{2}\nu^1\right)dt + h^1\partial_\zeta\left(\frac{\Phi^2(\mathrm{y},\zeta)}{2}\nu^2\right)dt + \overline{h^2}\Phi(\mathrm{x},\xi)\nu^1 dW_t - h^1\Phi(\mathrm{y},\zeta)\nu^2 dW_t \\
&+ \overline{h^2}\partial_\xi dm_1(t,\mathrm{x},\xi) - h^1\partial_\zeta dm_2(t,\mathrm{y},\zeta)dt.
\end{aligned} \tag{40}$$

Inserting (39) into (40), we get

$$\begin{aligned}
&d(h^1\overline{h^2}) + \Phi(\mathrm{x},\xi)\Phi(\mathrm{y},\zeta)\nu^1\otimes\nu^2 dt + \overline{h^2}\mathfrak{f}'(\mathrm{x},\xi)\cdot\nabla_{g,\mathrm{x}} h^1 dt + h^1\mathfrak{f}'(\mathrm{y},\zeta)\cdot\nabla_{g,\mathrm{y}}\overline{h^2} dt \\
&= -\overline{h^2}\partial_\xi\left(\frac{\Phi^2(\mathrm{x},\xi)}{2}\nu^1\right)dt + h^1\partial_\zeta\left(\frac{\Phi^2(\mathrm{y},\zeta)}{2}\nu^2\right)dt + (\overline{h^2}\Phi(\mathrm{x},\xi)\nu^1 - h^1\Phi(\mathrm{y},\zeta)\nu^2)dW_t \\
&+ \overline{h^2}\partial_\xi dm_1(t,\mathrm{x},\xi)dt - h^1\partial_\zeta dm_2(t,\mathrm{y},\zeta)dt.
\end{aligned} \tag{41}$$

We now choose the non-negative test function $\varphi(t,\mathrm{x},\mathrm{y},\xi,\zeta) = \rho(\mathrm{x}-\mathrm{y})\psi(\xi-\zeta)$, where ρ and ψ are smooth non-negative functions defined on appropriate Euclidean spaces. Multiplying (41) with φ and integrating over $(0,T)\times M^2\times\mathbb{R}^2$ we get

$$\begin{aligned}
&\int_{M^2}\int_{\mathbb{R}^2} h^1(T,\mathrm{x},\xi)\overline{h^2}(T,\mathrm{y},\zeta)\rho(\mathrm{x}-\mathrm{y})\psi(\xi-\zeta)d\zeta d\xi d\mathrm{y}d\mathrm{x} \qquad (42)\\
&-\int_{M^2}\int_{\mathbb{R}^2} h_0^1\overline{h_0^2}\rho(\mathrm{x}-\mathrm{y})\psi(\xi-\zeta)d\zeta d\xi d\mathrm{y}d\mathrm{x} \\
&+\int_0^T\int_{M^2}\int_{\mathbb{R}^2}\rho(\mathrm{x}-\mathrm{y})\psi(\xi-\zeta)\Phi(\mathrm{x},\xi)\Phi(\mathrm{y},\zeta)d\nu^2_{(t,\mathrm{y})}(\zeta)d\nu^1_{(t,\mathrm{x})}(\xi)d\mathrm{y}d\mathrm{x}dt \\
&+\int_0^T\int_{M^2}\int_{\mathbb{R}^2}\mathfrak{f}'(\mathrm{x},\xi)\cdot\nabla_{g,\mathrm{x}} h^1(t,\mathrm{x},\xi)\overline{h^2}(t,\mathrm{y},\zeta)\rho(\mathrm{x}-\mathrm{y})\psi(\xi-\zeta)d\zeta d\xi d\mathrm{y}d\mathrm{x}dt \\
&+\int_0^T\int_{M^2}\int_{\mathbb{R}^2}\mathfrak{f}'(\mathrm{y},\zeta)\cdot\nabla_{g,\mathrm{y}}\overline{h^2}(t,\mathrm{y},\zeta)h^1(t,\mathrm{x},\xi)\rho(\mathrm{x}-\mathrm{y})\psi(\xi-\zeta)d\zeta d\xi d\mathrm{y}d\mathrm{x}dt \\
&=\int_0^T\int_{M^2}\int_{\mathbb{R}^2}\frac{\Phi^2(\mathrm{x},\xi)}{2}\overline{h^2}(t,\mathrm{y},\zeta)\rho(\mathrm{x}-\mathrm{y})\psi'(\xi-\zeta)d\nu^1_{(t,\mathrm{x})}(\xi)d\zeta d\mathrm{y}d\mathrm{x}dt \\
&+\int_0^T\int_{M^2}\int_{\mathbb{R}^2}\frac{\Phi^2(\mathrm{y},\zeta)}{2}h^1(t,\mathrm{x},\xi)\rho(\mathrm{x}-\mathrm{y})\psi'(\xi-\zeta)d\nu^2_{(t,\mathrm{y})}(\zeta)d\xi d\mathrm{y}d\mathrm{x}dt \\
&+\int_0^T\int_{M^2}\int_{\mathbb{R}^2}\rho(\mathrm{x}-\mathrm{y})\psi(\xi-\zeta)\overline{h^2}(t,\mathrm{y},\zeta)\Phi(\mathrm{x},\xi)d\nu^1_{(t,\mathrm{x})}(\xi)d\zeta d\mathrm{y}d\mathrm{x}dW_t
\end{aligned}$$

$$- \int_0^T \int_{M^2} \int_{\mathbb{R}^2} \rho(x-y)\psi(\xi-\zeta) h^1(t,x,\xi)\Phi(y,\zeta) d\nu^2_{(t,y)}(\zeta) d\xi dy dx dW_t$$

$$- \int_0^T \int_{M^2} \int_{\mathbb{R}^2} \rho(x-y)\psi'(\xi-\zeta)\overline{h^2}(t,y,\zeta) dm_1(t,x,\xi) d\zeta dy$$

$$- \int_0^T \int_{M^2} \int_{\mathbb{R}^2} \rho(x-y)\psi'(\xi-\zeta) h^1(t,x,\xi) dm_2(t,y,\zeta) d\xi dx.$$

By using integration by parts with respect to ζ and ξ in the first and second and in the last two terms on the right hand side in (42), and using $\partial_\xi h^1 = -\nu^1$ and $\partial_\zeta \overline{h^2} = \nu^2$, we obtain:

$$\int_{M^2} \int_{\mathbb{R}^2} h^1(T,x,\xi)\overline{h^2}(T,y,\zeta)\rho(x-y)\psi(\xi-\zeta) d\zeta d\xi dy dx \tag{43}$$

$$- \int_{M^2} \int_{\mathbb{R}^2} h^1_0(x,\xi)\overline{h^2_0}(y,\zeta)\rho(x-y)\psi(\xi-\zeta) d\zeta d\xi dy dx$$

$$+ \int_0^T \int_{M^2} \int_{\mathbb{R}^2} \rho(x-y)\psi(\xi-\zeta)\Phi(x,\xi)\Phi(y,\zeta) d\nu^2_{(t,y)}(\zeta) d\nu^1_{(t,x)}(\xi) dy dx dt$$

$$+ \int_0^T \int_{M^2} \int_{\mathbb{R}^2} \mathfrak{f}'(x,\xi)\cdot\nabla_{g,x} h^1(t,x,\xi)\overline{h^2}(t,y,\zeta)\rho(x-y)\psi(\xi-\zeta) d\zeta d\xi dy dx dt$$

$$+ \int_0^T \int_{M^2} \int_{\mathbb{R}^2} \mathfrak{f}'(y,\zeta)\cdot\nabla_{g,y}\overline{h^2}(t,y,\zeta) h^1(t,x,\xi)\rho(x-y)\psi(\xi-\zeta) d\zeta d\xi dy dx dt$$

$$= \int_0^T \int_{M^2} \int_{\mathbb{R}^2} \frac{\Phi^2(x,\xi)}{2}\rho(x-y)\psi(\xi-\zeta) d\nu^2_{(t,y)}(\zeta) d\nu^1_{(t,x)}(\xi) dy dx dt$$

$$+ \int_0^T \int_{M^2} \int_{\mathbb{R}^2} \frac{\Phi^2(y,\zeta)}{2}\rho(x-y)\psi(\xi-\zeta) d\nu^2_{(t,y)}(\zeta) d\nu^1_{(t,x)}(\xi) dy dx dt$$

$$+ \int_0^T \int_{M^2} \int_{\mathbb{R}^2} \rho(x-y)\psi(\xi-\zeta)\overline{h^2}(t,y,\zeta)\Phi(x,\xi) d\nu^1_{(t,x)}(\xi) d\zeta dy dx dW_t$$

$$- \int_0^T \int_{M^2} \int_{\mathbb{R}^2} \rho(x-y)\psi(\xi-\zeta) h^1(t,x,\xi)\Phi(y,\zeta) d\nu^2_{(t,y)}(\zeta) d\xi dy dx dW_t$$

$$- \int_0^T \int_{M^2} \int_{\mathbb{R}^2} \rho(x-y)\psi(\xi-\zeta)\nu^2_{(t,y)}(\zeta) dm_1(t,x,\xi) d\zeta dy$$

$$- \int_0^T \int_{M^2} \int_{\mathbb{R}^2} \rho(x-y)\psi(\xi-\zeta)\nu^1_{(t,x)}(\xi) dm_2(t,y,\zeta) d\xi dx.$$

Finally, moving the third term on the left hand side in (43) to the right hand side and using non-negativity of the measures m_1 and m_2 yields

$$\int_{M^2}\int_{\mathbb{R}^2} h^1(T,x,\xi)\overline{h^2}(T,y,\zeta)\rho(x-y)\psi(\xi-\zeta)d\zeta d\xi dy dx \tag{44}$$
$$-\int_{M^2}\int_{\mathbb{R}^2} h^1_0(x,\xi)\overline{h^2_0}(y,\zeta)\rho(x-y)\psi(\xi-\zeta)d\zeta d\xi dy dx$$
$$+\int_0^T\int_{M^2}\int_{\mathbb{R}^2} f'(x,\xi)\cdot\nabla_{g,x}h^1(t,x,\xi)\overline{h^2}(t,y,\zeta)\rho(x-y)\psi(\xi-\zeta)d\zeta d\xi dy dx dt$$
$$+\int_0^T\int_{M^2}\int_{\mathbb{R}^2} f'(y,\zeta)\cdot\nabla_{g,y}\overline{h^2}(t,y,\zeta)h^1(t,x,\xi)\rho(x-y)\psi(\xi-\zeta)d\zeta d\xi dy dx dt$$
$$\leq\frac{1}{2}\int_0^T\int_{M^2}\int_{\mathbb{R}^2}(\Phi(x,\xi)-\Phi(y,\zeta))^2\rho(x-y)\psi(\xi-\zeta)d\nu^2_{(t,y)}(\zeta)d\nu^1_{(t,x)}(\xi)dy dx dt$$
$$+\int_0^T\int_{M^2}\int_{\mathbb{R}^2}\rho(x-y)\psi(\xi-\zeta)\overline{h^2}(t,y,\zeta)\Phi(x,\xi)d\nu^1_{(t,x)}(\xi)d\zeta dy dx dW_t$$
$$-\int_0^T\int_{M^2}\int_{\mathbb{R}^2}\rho(x-y)\psi(\xi-\zeta)h^1(t,x,\xi)\Phi(y,\zeta)d\nu^2_{(t,y)}(\zeta)d\xi dy dx dW_t.$$

Setting $\psi(\xi)=\delta(\xi)$ and $\rho(x)=\delta(x)$ and rearranging it a bit, we obtain

$$\int_M\int_{\mathbb{R}} h^1(T,x,\xi)\overline{h^2}(T,x,\xi)d\xi dx$$
$$\leq\int_M\int_{\mathbb{R}} h^1_0\overline{h^2_0}d\xi dx-\int_0^T\int_M\int_{\mathbb{R}} f'(x,\xi)\cdot\nabla_{g,x}(h^1(t,x,\xi)\overline{h^2}(t,x,\xi))d\xi dx dt$$
$$-\int_0^T\int_M\int_{\mathbb{R}}\Phi(x,\xi)\partial_\xi(h^1(t,x,\xi)\overline{h^2}(t,x,\xi))d\xi dx dW_t. \tag{45}$$

Another integration by parts provides

$$\int_M\int_{\mathbb{R}} h^1(T,x,\xi)\overline{h^2}(T,x,\xi)d\xi dx \tag{46}$$
$$\leq\int_M\int_{\mathbb{R}} h^1_0\overline{h^2_0}d\xi dx+\int_0^T\int_M\int_{\mathbb{R}}\Phi'(x,\xi)h^1(t,x,\xi)\overline{h^2}(t,x,\xi)d\xi dx dW(t)$$

where we used the geometry compatibility conditions to eliminate the flux term.

By using non-negativity of h^1 and $\overline{h^2}$, we have after finding expectation of square of (46) and taking into account the Itô isometry

$$E\left[\Big(\int_M\int_{\mathbb{R}} h^1(T,x,\xi)\overline{h^2}(T,x,\xi)d\xi dx\Big)^2\right] \tag{47}$$
$$\lesssim E\left[\Big(\int_M\int_{\mathbb{R}} h_0^1\overline{h_0^2}d\xi dx\Big)^2\right] + \|\Phi'\|_\infty^2 E\left[\int_0^T\Big(\int_M\int_{\mathbb{R}} h^1(t,x,\xi)\overline{h^2}(t,x,\xi)d\xi dx\Big)^2 dt\right].$$

From here, using the Gronwall inequality, we get

$$E\left[\Big(\int_M\int_{\mathbb{R}} h^1(T,x,\xi)\overline{h^2}(T,x,\xi)d\xi dx\Big)^2\right] \lesssim E\left[\Big(\int_M |u_{10}(x)-u_{20}(x)|dx\Big)^2\right]. \tag{48}$$

From here, if assume that $u_{10}=u_{20}$, we get almost surely for almost every $(t,x,\xi)\in[0,\infty)\times M\times\mathbb{R}$:

$$h^1(t,x,\xi)\,(1-h^2(t,x,\xi))=0.$$

This implies that either $h^1(t,x,\xi)=0$ or $h^2(t,x,\xi)=1$. Since we can interchange the roles of h^1 and h^2, we conclude that 1 and 0 are actually the only values that h^1 or h^2 can attain and that $h^1=h^2=h$. Since h is also non-increasing with respect to ξ on $[0,\infty)$, we conclude (taking into account the initial data $h_0=\mathrm{sign}_+(u_0(x)-\xi)$) that there exists a function $u:[0,\infty)\times M\to\mathbb{R}$ such that

$$h(t,x,\xi)=\mathrm{sign}_+(u(t,x)-\xi). \tag{49}$$

We thus have the following corollary which is proven in the final section.

Corollary 4.1. *The generalized stochastic kinetic solution to* (1), (2) *has the form* (49). *If the function u satisfies the second item from Definition 3.3, then it is an admissible stochastic solution to* (1), (2).

5. Uniqueness – rigorous proof

In this section, we shall formalize the arguments from the previous section. To this end, it will be necessary to express (35) in local coordinates. So, assume we are given a generalized stochastic kinetic solution h. To prove uniqueness locally we take a chart (U,κ) for M and assume, without loss of generality, that $\kappa(U)=\mathbb{R}^d$. Define the local expression of h as the map (in order to avoid proliferation of symbols, we shall keep the same notations for global and local quantities but we shall write $\tilde{x}$ to denote the local variable)

$$h:\mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R}\times\Omega\to\mathbb{R},\quad h(t,\tilde{x},\xi,\omega)=h(t,\kappa^{-1}(\tilde{x}),\xi,\omega)G(\tilde{x}),$$

where $G(\tilde{x})$ is the Gramian corresponding to the chart (U,κ). Similarly, for $\tilde{x}\in\mathbb{R}^d$ we define

$$\begin{aligned}\Phi(\tilde{x},\xi)&=\Phi(\kappa^{-1}(\tilde{x}),\xi),\\ \mathfrak{f}(\tilde{x},\xi)&=\mathfrak{f}(\kappa^{-1}(\tilde{x}),\xi),\quad \mathfrak{f}'(\tilde{x},\xi)=\mathfrak{f}'(\kappa^{-1}(\tilde{x}),\xi)=a(\tilde{x},\xi)\\ \nu_{(t,\tilde{x})}(\lambda)&=\nu_{(t,\kappa^{-1}(\tilde{x}))}(\lambda)G(\tilde{x}),\end{aligned} \tag{50}$$

and $m(t,\tilde{x},\xi)$ will be the pushforward measure of m with respect to the mapping κ.

With such notations at hand, we now rewrite (35) locally in the chart (U,κ) into an equation in terms of $h_1(t,\tilde{x},\xi)$ and $\overline{h_2}(t,\tilde{x},\xi)$, which are two generalized kinetic solutions to Cauchy problems corresponding to (1) with the initial data u_{10} and u_{20}, respectively. Below, we use the Einstein summation convention and we remind that $a=(a_1,\dots,a_d)=\mathfrak{f}'=(f_1',\dots,f_d')$. Also, since the equations are to be understood in the weak sense, we need to add the Gramian in each of the terms below except in m_1 and m_2, since the corresponding part in these terms is implied there by the definition of the pushforward measure. This is why we introduce the conventions from (50).

$$dh^1(t,\tilde{x},\xi)+\operatorname{div}_{\tilde{x}}(a(\tilde{x},\xi)h^1)dt+h^1\Gamma^j_{kj}(\tilde{x})a_k(t,\tilde{x},\xi)dt \tag{51}$$
$$=-\partial_\xi\left(\frac{\Phi^2(\tilde{x},\xi)}{2}\nu^1_{(t,\tilde{x})}(\xi)\right)dt+\Phi(\tilde{x},\xi)\nu^1_{(t,\tilde{x})}(\xi)dW_t+\partial_\xi dm_1,$$
$$d\overline{h^2}(t,\tilde{y},\zeta)+\operatorname{div}_{\tilde{y}}(a(\tilde{y},\zeta)\overline{h^2})dt+\overline{h^2}\Gamma^j_{kj}(\tilde{y})a_k(t,\tilde{y},\zeta)dt \tag{52}$$
$$=\partial_\zeta\left(\frac{\Phi^2(\tilde{y},\zeta)}{2}\nu^2_{(t,\tilde{y})}(\zeta)\right)dt-\Phi(\tilde{y},\zeta)\nu^2_{(t,\tilde{y})}(\zeta)dW_t-\partial_\zeta dm_2$$

We introduce two mollifying functions $\omega_1\in\mathcal{D}(\mathbb{R}^d)$, $\omega_2\in\mathcal{D}(\mathbb{R})$ where d is the dimension of the manifold M, such that $\omega_i\geq 0$, $i=1,2$ and $\int_{\mathbb{R}^d}\omega_1=\int_{\mathbb{R}}\omega_2=1$. Taking $\omega_{\delta,r}(\tilde{x},\xi,t)=\frac{1}{r\delta^d}\omega_1\left(\frac{\tilde{x}}{\delta}\right)\omega_2\left(\frac{\xi}{r}\right)$, for some $\delta,r>0$, and using convolution, (51) and (52) yield (below and in the sequel, subscripts δ and r denote convolution with respect to the corresponding variables):

$$dh^1_{\delta,r}+\operatorname{div}_{\tilde{x}}(a(\tilde{x},\xi)h^1_{\delta,r})dt+g^1_{\delta,r}dt+\left(\Gamma^j_{kj}(\tilde{x})a_k(t,\tilde{x},\xi)h^1\right)_{\delta,r}dt \tag{53}$$
$$=-\partial_\xi\left(\frac{\Phi^2(\tilde{x},\xi)}{2}\nu^1_{(t,\tilde{x})}(\xi)dt\right)_{\delta,r}+(\Phi(\tilde{x},\xi)\nu^1_{(t,\tilde{x})}(\xi))_{\delta,r}dW_t+\partial_\xi dm_{1,\delta,r},$$
$$d\overline{h^2_{\delta,r}}+\operatorname{div}_{\tilde{y}}(a(\tilde{y},\zeta)\overline{h^2_{\delta,r}})dt+g^2_{\delta,r}dt+\left(\Gamma^j_{kj}(\tilde{y})a_k(t,\tilde{y},\zeta)\overline{h^2}\right)_{\delta,r}dt \tag{54}$$
$$=\partial_\zeta\left(\frac{\Phi^2(\tilde{y},\zeta)}{2}\nu^1_{(t,\tilde{y})}(\zeta)dt\right)_{\delta,r}-(\Phi(\tilde{y},\zeta)\nu^1_{(t,\tilde{y})}(\zeta))_{\delta,r}dW_t-\partial_\zeta dm_{2,\delta,r}$$

where

$$g^1_{\delta,r}=\operatorname{div}_{\tilde{x}}(a(\tilde{x},\xi)h^1)_{\delta,r}-\operatorname{div}_{\tilde{x}}(a(\tilde{x},\xi)h^1_{\delta,r})$$
$$g^2_{\delta,r}=\operatorname{div}_{\tilde{y}}(a(\tilde{y},\zeta)\overline{h^2})_{\delta,r}-\operatorname{div}_{\tilde{y}}(a(\tilde{y},\zeta)\overline{h^2_{\delta,r}}).$$

These terms converge to zero as $\delta,r\to 0$ according to the Friedrichs lemma [30].

Now, multiplying (53) and (54) with $\overline{h^2_{\delta,r}}=\overline{h^2_{\delta,r}}(t,\mathrm{y},\zeta)$ and $h^1_{\delta,r}=h^1_{\delta,r}(t,\mathrm{x},\xi)$, respectively, and using (26), we obtain

$$d(h^1_{\delta,r}\overline{h^2_{\delta,r}})+(\Phi(\tilde{x},\xi)\nu^1_{(t,\tilde{x})}(\xi))_{\delta,r}(\Phi(\tilde{y},\zeta)\nu^2_{(t,y)}(\zeta))_{\delta,r}dt \tag{55}$$
$$+\overline{h^2_{\delta,r}}\operatorname{div}_{\tilde{x}}(a(\tilde{x},\xi)h^1_{\delta,r})dt+h^1_{\delta,r}\operatorname{div}_{\tilde{y}}(a(\tilde{y},\zeta)\overline{h^2_{\delta,r}})dt$$
$$+\left(\Gamma^j_{kj}(\tilde{x})a_k(t,\tilde{x},\xi)h^1\right)_{\delta,r}\overline{h^2_{\delta,r}}dt+\left(\Gamma^j_{kj}(\tilde{y})a_k(t,\tilde{y},\zeta)\overline{h^2}\right)_{\delta,r}h^1_{\delta,r}dt=$$
$$-g^1_{\delta,r}\overline{h^2_{\delta,r}}dt-g^2_{\delta,r}h^1_{\delta,r}dt+\overline{h^2_{\delta,r}}(\Phi(\tilde{x},\xi)\nu^1_{(t,\tilde{x})}(\xi))_{\delta,r}dW_t-h^1_{\delta,r}(\Phi(\tilde{y},\zeta)\nu^2_{(t,\tilde{y})}(\zeta))_{\delta,r}dW_t$$
$$+h^1_{\delta,r}\partial_\zeta\left(\frac{\Phi^2(\tilde{y},\zeta)}{2}\nu^2_{(t,\tilde{y})}(\zeta)\right)_{\delta,r}dt-\overline{h^2_{\delta,r}}\partial_\xi\left(\frac{\Phi^2(\tilde{x},\xi)}{2}\nu^1_{(t,\tilde{x})}(\xi)\right)_{\delta,r}dt$$
$$+\overline{h^2_{\delta,r}}\partial_\xi dm_{1,\delta,r}(t,\tilde{x},\xi)dt-h^1_{\delta,r}\partial_\zeta dm_{2,\delta,r}(t,\tilde{y},\zeta)dt.$$

Next, we choose non-negative functions $\rho\in\mathcal{D}(\mathbb{R}^d)$, $\psi,\varphi\in\mathcal{D}(\mathbb{R})$ such that $\int_{\mathbb{R}^d}\rho=\int_{\mathbb{R}}\psi=1$. Using the test function $\rho_\varepsilon(\tilde{x}-\tilde{y})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{x}+\tilde{y}}{2}\right)$, with $\rho_\varepsilon(\tilde{x})=\frac{1}{\varepsilon^d}\rho\left(\frac{\tilde{x}}{\varepsilon}\right)$, $\psi_\varepsilon(\xi)=\frac{1}{\varepsilon}\psi\left(\frac{\xi}{\varepsilon}\right)$, for some $\varepsilon>0$, and integrating

(55) over $(0,T)$, the equation is rewritten in the variational formulation (recall that h^1 and h^2 are continuous with respect to $t \in \mathbb{R}^+$):

$$\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2} h^1_{\delta,r}(T,\tilde{x},\xi)\overline{h^2_{\delta,r}}(T,\tilde{y},\zeta)\rho_\varepsilon(\tilde{x}-\tilde{y})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{x}+\tilde{y}}{2}\right)d\xi d\zeta d\tilde{x}d\tilde{y} \tag{56}$$

$$-\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2} h^1_{0,\delta,r}(\tilde{x},\xi)\overline{h^2_{0,\delta,r}}(\tilde{y},\zeta)\rho_\varepsilon(\tilde{x}-\tilde{y})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{x}+\tilde{y}}{2}\right)d\xi d\zeta d\tilde{x}d\tilde{y} \tag{57}$$

$$+\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}\left(\overline{h^2_{\delta,r}}\,\mathrm{div}_{\tilde{x}}(a(\tilde{x},\xi)h^1_{\delta,r}) + h^1_{\delta,r}\,\mathrm{div}_{\tilde{y}}(a(\tilde{y},\zeta)\overline{h^2_{\delta,r}})\right)\times \tag{58}$$

$$\times\,\rho_\varepsilon(\tilde{x}-\tilde{y})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{x}+\tilde{y}}{2}\right)d\xi d\zeta d\tilde{x}d\tilde{y}dW_t$$

$$+\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}\left(\left(\Gamma^j_{kj}(\tilde{x})a_k(t,\tilde{x},\xi)h^1\right)_{\delta,r}\overline{h^2_{\delta,r}} + \left(\Gamma^j_{kj}(\tilde{y})a_k(t,\tilde{y},\zeta)\overline{h^2}\right)_{\delta,r}h^1_{\delta,r}\right)\times \tag{59}$$

$$\times\,\rho_\varepsilon(\tilde{x}-\tilde{y})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{x}+\tilde{y}}{2}\right)d\xi d\zeta d\tilde{x}d\tilde{y}$$

$$=-\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}\left(g^1_{\delta,r}\overline{h^2_{\delta,r}} + g^2_{\delta,r}h^1_{\delta,r} - \overline{h^2_{\delta,r}}(\Phi(\tilde{x},\xi)\nu^1_{(t,\tilde{x})}(\xi))_{\delta,r} + h^1_{\delta,r}(\Phi(\tilde{y},\zeta)\nu^2_{(t,\tilde{y})}(\zeta))_{\delta,r}\right)\times \tag{60}$$

$$\times\,\rho_\varepsilon(\tilde{x}-\tilde{y})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{x}+\tilde{y}}{2}\right)d\xi d\zeta d\tilde{x}d\tilde{y}dW_t$$

$$+\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}\left(h^1_{\delta,r}\partial_\zeta\left(\frac{\Phi^2(\tilde{y},\zeta)}{2}\nu^2_{(t,\tilde{y})}(\zeta)\right)_{\delta,r} - \overline{h^2_{\delta,r}}\partial_\xi\left(\frac{\Phi^2(\tilde{x},\xi)}{2}\nu^1_{(t,\tilde{x})}(\xi)\right)_{\delta,r}\right. \tag{61}$$

$$\left.-\,(\Phi(\tilde{x},\xi)\nu^1_{(t,\tilde{x})}(\xi))_{\delta,r}(\Phi(\tilde{y},\zeta)\nu^2_{(t,y)}(\zeta))_{\delta,r}\right)\rho_\varepsilon(\tilde{x}-\tilde{y})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{x}+\tilde{y}}{2}\right)d\xi d\zeta d\tilde{x}d\tilde{y}dt$$

$$+\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}\left(\overline{h^2_{\delta,r}}(t,\tilde{y},\zeta)\partial_\xi m_{1,\delta,r}(t,\tilde{x},\xi) - h^1_{\delta,r}(t,\tilde{y},\xi)\partial_\zeta m_{2,\delta,r}(t,\tilde{y},\zeta)\right)\times \tag{62}$$

$$\times\,\rho_\varepsilon(\tilde{x}-\tilde{y})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{x}+\tilde{y}}{2}\right)d\xi d\zeta d\tilde{x}d\tilde{y}dt.$$

We shall analyze this equality term by term. We start with the terms from (56)–(58). We have:

$$\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2} h^1_{\delta,r}(T,\tilde{x},\xi)\overline{h^2_{\delta,r}}(T,\tilde{y},\zeta)\rho_\varepsilon(\tilde{x}-\tilde{y})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{x}+\tilde{y}}{2}\right)d\zeta d\xi d\tilde{y}d\tilde{x} \tag{63}$$

$$-\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2} h^1_{\delta,r}(0,\tilde{x},\xi)\overline{h^2_{\delta,r}}(0,\tilde{y},\zeta)\rho_\varepsilon(\tilde{x}-\tilde{y})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{x}+\tilde{y}}{2}\right)d\zeta d\xi d\tilde{y}d\tilde{x}$$

$$-\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2} a(\tilde{x},\xi)h^1_{\delta,r}(t,\tilde{x},\xi)\overline{h^2_{\delta,r}}(t,\tilde{y},\zeta)\cdot\left[\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{x}+\tilde{y}}{2}\right)\nabla\rho_\varepsilon(\tilde{x}-\tilde{y})\right.$$

$$\left.+\,\frac{1}{2}\rho_\varepsilon(\tilde{x}-\tilde{y})\psi_\varepsilon(\xi-\zeta)\nabla\varphi\left(\frac{\tilde{x}+\tilde{y}}{2}\right)\right]d\zeta d\xi d\tilde{y}d\tilde{x}dt$$

$$+\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2} a(\tilde{\mathbf{y}},\zeta)h^1_{\delta,r}(t,\tilde{\mathbf{x}},\xi)\overline{h^2_{\delta,r}}(t,\tilde{\mathbf{y}},\zeta)\cdot\Big[\psi_\varepsilon(\xi-\zeta)\varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big)\nabla\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})$$
$$-\frac{1}{2}\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})\psi_\varepsilon(\xi-\zeta)\nabla\varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big)\Big]d\zeta d\xi d\tilde{\mathbf{y}}d\tilde{\mathbf{x}}dt$$
$$=\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2} h^1_{\delta,r}(T,\tilde{\mathbf{x}},\xi)\overline{h^2_{\delta,r}}(T,\tilde{\mathbf{y}},\zeta)\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})\psi_\varepsilon(\xi-\zeta)\varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big)d\zeta d\xi d\tilde{\mathbf{y}}d\tilde{\mathbf{x}}-$$
$$-\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2} h^1_{\delta,r}(0,\tilde{\mathbf{x}},\xi)\overline{h^2_{\delta,r}}(0,\tilde{\mathbf{y}},\zeta)\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})\psi_\varepsilon(\xi-\zeta)\varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big)d\zeta d\xi d\tilde{\mathbf{y}}d\tilde{\mathbf{x}}$$
$$-\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}(a(\tilde{\mathbf{x}},\xi)-a(\tilde{\mathbf{y}},\zeta))\cdot\nabla\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})h^1_{\delta,r}(t,\tilde{\mathbf{x}},\xi)\overline{h^2_{\delta,r}}(t,\tilde{\mathbf{y}},\zeta)\psi_\varepsilon(\xi-\zeta)\varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big)d\zeta d\xi d\tilde{\mathbf{y}}d\tilde{\mathbf{x}}dt$$
$$-\frac{1}{2}\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}(a(\tilde{\mathbf{x}},\xi)+a(\tilde{\mathbf{y}},\zeta))\cdot\nabla\varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big)h^1_{\delta,r}(t,\tilde{\mathbf{x}},\xi)\overline{h^2_{\delta,r}}(t,\tilde{\mathbf{y}},\zeta)\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})\psi_\varepsilon(\xi-\zeta)d\zeta d\xi d\tilde{\mathbf{y}}d\tilde{\mathbf{x}}dt$$

The penultimate term in (63) can be rewritten as (below $dV=d\zeta d\xi d\tilde{\mathbf{y}}d\tilde{\mathbf{x}}dt$):

$$\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}(a(\tilde{\mathbf{x}},\xi)-a(\tilde{\mathbf{y}},\zeta))\cdot\nabla\rho_\varepsilon(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})h^1_{\delta,r}(t,\tilde{\mathbf{x}},\xi)\overline{h^2_{\delta,r}}(t,\tilde{\mathbf{y}},\zeta)\psi_\varepsilon(\xi-\zeta)\varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big)dV= \tag{64}$$
$$\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}(a(\tilde{\mathbf{x}},\xi)-a(\tilde{\mathbf{y}},\zeta))\cdot\nabla\Big(\frac{1}{\varepsilon^d}\rho\Big(\frac{\tilde{\mathbf{x}}-\tilde{\mathbf{y}}}{\varepsilon}\Big)\Big)h^1_{\delta,r}(t,\tilde{\mathbf{x}},\xi)\overline{h^2_{\delta,r}}(t,\tilde{\mathbf{y}},\zeta)\psi_\varepsilon(\xi-\zeta)\varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big)dV=$$
$$\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}(a(\tilde{\mathbf{x}},\xi)-a(\tilde{\mathbf{y}},\zeta))\cdot\frac{1}{\varepsilon^d}\nabla\rho(\mathbf{z})\Big|_{\mathbf{z}=\frac{\tilde{\mathbf{x}}-\tilde{\mathbf{y}}}{\varepsilon}}h^1_{\delta,r}(t,\tilde{\mathbf{x}},\xi)\overline{h^2_{\delta,r}}(t,\tilde{\mathbf{y}},\zeta)\psi_\varepsilon(\xi-\zeta)\varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big)dV=$$
$$\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}\frac{a(\tilde{\mathbf{x}},\xi)-a(\tilde{\mathbf{y}},\zeta)}{\varepsilon}\cdot\frac{1}{\varepsilon^d}\nabla\rho(\mathbf{z})\Big|_{\mathbf{z}=\frac{\tilde{\mathbf{x}}-\tilde{\mathbf{y}}}{\varepsilon}}h^1_{\delta,r}(t,\tilde{\mathbf{x}},\xi)\overline{h^2_{\delta,r}}(t,\tilde{\mathbf{y}},\zeta)\psi_\varepsilon(\xi-\zeta)\varphi\Big(\frac{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}}{2}\Big)dV=$$
$$\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}\frac{a_k(\varepsilon\mathbf{z}+\tilde{\mathbf{y}},\xi)-a_k(\tilde{\mathbf{y}},\zeta)}{\varepsilon z_k}z_k\partial_{z_k}\rho(\mathbf{z})h^1_{\delta,r}(t,\tilde{\mathbf{y}}+\varepsilon\mathbf{z},\xi)\overline{h^2_{\delta,r}}(t,\tilde{\mathbf{y}},\zeta)\psi_\varepsilon(\xi-\zeta)\varphi\Big(\tilde{\mathbf{y}}+\frac{\varepsilon\mathbf{z}}{2}\Big)dV$$

where $\mathbf{z}=\frac{\tilde{\mathbf{x}}-\tilde{\mathbf{y}}}{\varepsilon}$. We notice that, as $r,\delta,\varepsilon\to 0$ (in any order), this term becomes

$$\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}}\partial_{\tilde{y}_k}a_k(\tilde{\mathbf{y}},\xi)h^1(t,\tilde{\mathbf{y}},\xi)\overline{h^2}(t,\tilde{\mathbf{y}},\xi)\varphi(\tilde{\mathbf{y}})\int_{\mathbb{R}^d}z_k\partial_{z_k}\rho(\mathbf{z})d\mathbf{z}\,d\xi d\tilde{\mathbf{y}}dt \tag{65}$$
$$=-\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}}\mathrm{div}_{\tilde{\mathbf{y}}}\,a(\tilde{\mathbf{y}},\xi)h^1(t,\tilde{\mathbf{y}},\xi)\overline{h^2}(t,\tilde{\mathbf{y}},\xi)\varphi(\tilde{\mathbf{y}})d\xi d\tilde{\mathbf{y}}dt$$
$$\overset{(3)}{=}\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}}\Gamma^j_{kj}(\tilde{\mathbf{y}})a_k(t,\tilde{\mathbf{y}},\xi)h^1(t,\tilde{\mathbf{y}},\xi)\overline{h^2}(t,\tilde{\mathbf{y}},\xi)d\xi d\tilde{\mathbf{y}}dt. \tag{66}$$

due to properties of the mollifier ρ. Thus, from (65) and (63) we conclude that as $r, \delta, \varepsilon \to 0$ in any order

$$(56)+(57)+(58) \underset{r,\delta,\varepsilon\to 0}{\longrightarrow} \tag{67}$$
$$\int_{\mathbb{R}^d}\int_{\mathbb{R}}(h^1\overline{h^2})(T,\tilde{y},\xi)\varphi(\tilde{x})d\xi d\tilde{x} - \int_{\mathbb{R}^d}\int_{\mathbb{R}}(h_0^1\overline{h_0^2})(\tilde{x},\xi)\varphi(\tilde{x})d\xi d\tilde{x}$$
$$-\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}} a(\tilde{x},\xi)(h^1\overline{h^2})(t,\tilde{x},\xi)\nabla\varphi(\tilde{x})d\xi d\tilde{x}dt - \int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}}\Gamma^j_{kj}(\tilde{x})a_k(t,\tilde{x},\xi)h^1(t,\tilde{x},\xi)\overline{h^2}(t,\tilde{x},\xi)d\xi d\tilde{x}dt.$$

Term (59) is easy to handle. We simply let $r, \delta, \varepsilon \to 0$ to conclude

$$(59) \underset{r,\delta,\varepsilon\to 0}{\longrightarrow} \int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}} 2\Gamma^j_{kj}(\tilde{x})a_k(t,\tilde{x},\xi)h^1\overline{h^2}\varphi(\tilde{x})d\xi d\tilde{x}dt. \tag{68}$$

In order to prepare handling (60) and (61), we use regularity of the function Φ (recall that $\Phi \in C^1_0(\mathbb{R}^d\times\mathbb{R})$). We have

$$\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}\Big[\Big(\frac{\Phi^2(\tilde{x},\xi)}{2}\nu^1_{(t,\tilde{x})}(\xi)\Big)_{\delta,r}\nu^2_{(t,\tilde{y}),\delta,r}(\zeta) - \frac{\Phi^2(\tilde{x},\xi)}{2}\nu^1_{(t,\tilde{x}),\delta,r}(\xi)\nu^2_{(t,\tilde{y}),\delta,r}(\zeta)\Big]\times \tag{69}$$
$$\times\rho_\varepsilon(\tilde{x}-\tilde{y})\psi_\varepsilon(\xi-\zeta)\varphi\Big(\frac{\tilde{x}+\tilde{y}}{2}\Big)d\zeta d\xi d\tilde{y}d\tilde{x} \underset{r,\delta\to 0}{\longrightarrow} 0,$$

and similarly

$$\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}\Big[\big(\Phi(\tilde{x},\xi)\nu^1_{(t,\tilde{x})}(\xi)\big)_{\delta,r}\big(\Phi(\tilde{y},\zeta)\nu^2_{(t,\tilde{y})}(\zeta)\big)_{\delta,r} - \big(\Phi(\tilde{x},\xi)\nu^1_{(t,\tilde{x}),\delta,r}(\xi)\big)\big(\Phi(\tilde{y},\zeta)\nu^2_{(t,\tilde{y}),\delta,r}(\zeta)\big)\Big]\times \tag{70}$$
$$\times\rho_\varepsilon(\tilde{x}-\tilde{y})\psi_\varepsilon(\xi-\zeta)\varphi\Big(\frac{\tilde{x}+\tilde{y}}{2}\Big)d\zeta d\xi d\tilde{y}d\tilde{x}dt \underset{r,\delta\to 0}{\longrightarrow} 0.$$

In a similar fashion, we have

$$\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}\Big[\big(\Phi(\tilde{x},\xi)\nu^1_{(t,\tilde{x})}(\xi)\big)_{\delta,r}\overline{h^2_{\delta,r}}(t,\tilde{y},\zeta) - \big(\Phi(\tilde{y},\zeta)\nu^2_{(t,\tilde{y})}(\zeta)\big)_{\delta,r}h^1_{\delta,r}(t,\tilde{x},\xi)\Big]\times$$
$$\times\rho_\varepsilon(\tilde{x}-\tilde{y})\psi_\varepsilon(\xi-\zeta)\varphi\Big(\frac{\tilde{x}+\tilde{y}}{2}\Big)d\zeta d\xi d\tilde{y}d\tilde{x}dW_t$$
$$=\int_0^T\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^2}\Big[\Phi(\tilde{x},\xi)\nu^1_{(t,\tilde{x}),\delta,r}(\xi)\overline{h^2_{\delta,r}}(t,\tilde{y},\zeta) - \Phi(\tilde{y},\zeta)\nu^2_{(t,\tilde{y}),\delta,r}(\zeta)h^1_{\delta,r}(t,\tilde{x},\xi)\Big]\times$$
$$\times\rho_\varepsilon(\tilde{x}-\tilde{y})\psi_\varepsilon(\xi-\zeta)\varphi\Big(\frac{\tilde{x}+\tilde{y}}{2}\Big)d\zeta d\xi d\tilde{y}d\tilde{x}dW_t + \int_0^T g_{3,\delta,r,\varepsilon}dW_t$$

where

$$g_{3,\delta,r} = \int\limits_{\mathbb{R}^{2d}}\int\limits_{\mathbb{R}^2}\Big[\Big(\big(\Phi(\tilde{x},\xi)\nu^1_{(t,\tilde{x})}(\xi)\big)_{\delta,r} - \Phi(\tilde{x},\xi)\big(\nu^1_{(t,\tilde{x})}(\xi)\big)_{\delta,r}\Big)\overline{h^2_{\delta,r}}(t,\tilde{y},\zeta)$$

$$-\Big(\big(\Phi(\tilde{y},\zeta)\nu^2_{(t,\tilde{y})}(\zeta)\big)_{\delta,r} - \Phi(\tilde{y},\zeta)\big(\nu^2_{(t,\tilde{y})}(\zeta)\big)_{\delta,r}\Big)h^1_{\delta,r}(t,\tilde{x},\xi)\Big]\times$$

$$\times\rho_\varepsilon(\tilde{x}-\tilde{y})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{x}+\tilde{y}}{2}\right)d\zeta d\xi d\tilde{y}d\tilde{x}$$

and $g_{3,\delta,r,\varepsilon}\to 0$ as $\delta,r\to 0$ almost surely. From here, using $\frac{dh^1_{\delta,r}(t,\tilde{x},\xi)}{\partial\xi} = -\nu^1_{(t,\tilde{x}),\delta,r}(\xi)$ and $\frac{d\overline{h^2_{\delta,r}}(t,\tilde{y},\zeta)}{\partial\zeta} = \nu^2_{(t,\tilde{y}),\delta,r}(\zeta)$, integration by parts, we have the following conclusion for (60)

$$\int\limits_0^T\int\limits_{\mathbb{R}^{2d}}\int\limits_{\mathbb{R}^2}\Big[\big(\Phi(\tilde{x},\xi)\nu^1_{(t,\tilde{x})}(\xi)\big)_{\delta,r}\overline{h^2_{\delta,r}}(t,\tilde{y},\zeta) - \big(\Phi(\tilde{y},\zeta)\nu^2_{(t,\tilde{y})}(\zeta)\big)_{\delta,r}h^1_{\delta,r}(t,\tilde{x},\xi)\Big]\times \tag{71}$$

$$\times\rho_\varepsilon(\tilde{x}-\tilde{y})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{x}+\tilde{y}}{2}\right)d\zeta d\xi d\tilde{y}d\tilde{x}dW_t$$

$$\underset{\varepsilon,r,\delta\to 0}{\longrightarrow}\int\limits_0^T\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}}\Phi'(\tilde{x},\xi)h^1(t,\tilde{x},\xi)\overline{h^2}(t,\tilde{x},\xi)\varphi(\tilde{x})\,d\xi d\tilde{x}dW_t$$

where we used the procedure leading to (65).

Having in mind (69), (70), and (71), we conclude that (61) has the following asymptotics:

$$\int\limits_0^T\int\limits_{\mathbb{R}^{2d}}\int\limits_{\mathbb{R}^2}\left(\frac{\Phi^2(\tilde{x},\xi)}{2}\nu^1_{(t,\tilde{x})}(\xi)\right)_{\delta,r}\overline{h^2_{\delta,r}}(t,\tilde{y},\zeta)\rho_\varepsilon(\tilde{x}-\tilde{y})\psi'_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{x}+\tilde{y}}{2}\right)d\zeta d\xi d\tilde{y}d\tilde{x}dt+ \tag{72}$$

$$+\int\limits_0^T\int\limits_{\mathbb{R}^{2d}}\int\limits_{\mathbb{R}^2}\left(\frac{\Phi^2(\tilde{y},\zeta)}{2}\nu^2_{(t,\tilde{y})}(\zeta)\right)_{\delta,r}h^1_{\delta,r}(t,\tilde{x},\xi)\rho_\varepsilon(\tilde{x}-\tilde{y})\psi'_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{x}+\tilde{y}}{2}\right)d\zeta d\xi d\tilde{y}d\tilde{x}dt-$$

$$-\int\limits_0^T\int\limits_{\mathbb{R}^{2d}}\int\limits_{\mathbb{R}^2}\big(\Phi(\tilde{x},\xi)\nu^1_{(t,\tilde{x})}(\xi)\big)_{\delta,r}\big(\Phi(\tilde{y},\zeta)\nu^2_{(t,\tilde{y})}(\zeta)\big)_{\delta,r}\times$$

$$\times\rho_\varepsilon(\tilde{x}-\tilde{y})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{x}+\tilde{y}}{2}\right)d\zeta d\xi d\tilde{y}d\tilde{x}dt-$$

$$-\int\limits_0^T\int\limits_{\mathbb{R}^{2d}}\int\limits_{\mathbb{R}^2}\Big(\big(\Phi(\tilde{x},\xi)\nu^1_{(t,\tilde{x})}(\xi)\big)_{\delta,r}\overline{h^2_{\delta,r}}(t,\tilde{y},\zeta) - \big(\Phi(\tilde{y},\zeta)\nu^2_{(t,\tilde{y})}(\zeta)\big)_{\delta,r}h^1_{\delta,r}(t,\tilde{x},\xi)\Big)\times$$

$$\times\rho_\varepsilon(\tilde{x}-\tilde{y})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{x}+\tilde{y}}{2}\right)d\zeta d\xi d\tilde{y}d\tilde{x}dW_t\underset{\varepsilon,r,\delta\to 0}{\longrightarrow}$$

$$\lim_{\varepsilon\to 0}\frac{1}{2}\int\limits_0^T\int\limits_{\mathbb{R}^{2d}}\int\limits_{\mathbb{R}^2}(\Phi(\tilde{x},\xi)-\Phi(\tilde{y},\zeta))^2\rho_\varepsilon(\tilde{x}-\tilde{y})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\tilde{x}+\tilde{y}}{2}\right)d\nu^2_{(t,y)}(\zeta)d\nu^1_{(t,x)}(\xi)dydxdt$$

$$-\int\limits_0^T\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}}\Phi'(\tilde{x},\xi)\varphi(\tilde{x})h^1(t,\tilde{x},\xi)\overline{h^2}(t,\tilde{x},\xi)d\xi d\tilde{x}dW_t$$

$$= -\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}} \Phi'(\bar{x},\xi)\varphi(\bar{x})h^1(t,\bar{x},\xi)\overline{h^2}(t,\bar{x},\xi)d\xi d\bar{x}dW_t. \tag{73}$$

Finally, we want to get rid of the entropy defect measures from (62). We use the fact that h^1 and h^2 are decreasing with respect to ξ (i.e. ζ) and that the measures m_1 and m_2 are non-negative. We have after two integration by parts (keep in mind that $\partial_\xi\psi(\xi-\zeta) = -\partial_\zeta\psi(\xi-\zeta)$)

$$\int_0^T \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} \Big(\overline{h^2_{\delta,r}}(t,\bar{y},\zeta)\partial_\xi m_{1,\delta,r}(t,\bar{x},\xi) - h^1_{\delta,r}(t,\bar{y},\xi)\partial_\zeta m_{2,\delta,r}(t,\bar{y},\zeta)\Big)\times \tag{74}$$
$$\times \rho_\varepsilon(\bar{x}-\bar{y})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\bar{x}+\bar{y}}{2}\right)d\xi d\zeta d\bar{x}d\bar{y}$$
$$= -\int_0^T \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} \Big(\nu^2_{(t,y),\varepsilon,\delta}(\zeta)m_{1,\delta,r}(t,\bar{x},\xi) + \nu^1_{(t,x),\varepsilon,\delta}(\xi)m_{2,\delta,r}(t,\bar{y},\zeta)\Big)\times$$
$$\times \rho_\varepsilon(\bar{x}-\bar{y})\psi_\varepsilon(\xi-\zeta)\varphi\left(\frac{\bar{x}+\bar{y}}{2}\right)d\xi d\zeta d\bar{x}d\bar{y} \leq 0.$$

Finally, from (67), (68), (71), (72), and (74), we conclude after letting $r,\delta,\varepsilon \to 0$ (first $r,\delta \to 0$ and then $\varepsilon \to 0$) that (56)–(62) becomes:

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}} h^1(T,\bar{x},\xi)\overline{h^2}(T,\bar{x},\xi)\varphi(\bar{x})d\xi d\bar{x} + \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}} \Gamma^j_{kj}(\bar{x})a_k(t,\bar{x},\xi)h^1(t,\bar{x},\xi)\overline{h^2}(t,\bar{x},\xi)\varphi(\bar{x})d\xi d\bar{x}dt$$
$$\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}} h^1_0\overline{h^2_0}\varphi(\bar{x})d\xi d\bar{x} + \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}} a(\bar{x},\xi)\cdot\nabla\varphi(\bar{x})(h^1\overline{h^2})(t,\bar{x},\xi)d\xi d\bar{x}dt$$
$$+ \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}} \Phi'(\bar{x},\xi)h^1(t,\bar{x},\xi)\overline{h^2}(t,\bar{x},\xi)\varphi(\bar{x})d\xi d\bar{x}dW_t.$$

From here, using the definition of the integral over a manifold and recalling (50), we see that it holds

$$\int_M \int_{\mathbb{R}} h^1(T,x,\xi)\overline{h^2}(T,x,\xi)G(\kappa(x))\varphi(x)d\xi dx \tag{75}$$
$$\leq \int_M \int_{\mathbb{R}} h^1_0(x,\xi)\overline{h^2_0}(x,\xi)G(\kappa(x))\varphi(x)d\xi dx - \int_0^T \int_M \int_{\mathbb{R}} (h^1\overline{h^2})(t,x,\xi)G(\kappa(x))a(x,\xi)\cdot\nabla_g\varphi(x)d\xi dxdt$$
$$+ \int_0^T \int_M \int_{\mathbb{R}} \Phi'(x,\xi)(h^1\overline{h^2})(t,x,\xi)G(\kappa(x))\varphi(x)d\xi dxdW_t.$$

Since we are on the compact manifold, we can take $\varphi \equiv 1$ which yields:

$$\int_M \int_{\mathbb{R}} h^1(T,x,\xi)\overline{h^2}(T,x,\xi)G(\kappa(x))d\xi dx \tag{76}$$
$$\leq \int_M \int_{\mathbb{R}} h_0^1(x,\xi)\overline{h_0^2}(x,\xi)G(\kappa(x))d\xi dx - \int_0^T \int_M \int_{\mathbb{R}} (h^1\overline{h^2})(t,x,\xi)G(\kappa(x))a(x,\xi)\cdot\nabla_g 1\, d\xi dx dt$$
$$+ \int_0^T \int_M \int_{\mathbb{R}} \Phi'(x,\xi)(h^1\overline{h^2})(t,x,\xi)G(\kappa(x))d\xi dx dW_t.$$

We arrived to (46) plus a term which does not affect using the Gronwall inequality and Itô isometry which give uniqueness as in (47). Remark that the Gramian has no influence on the procedure since it is a positive bounded function.

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ODLUKU
O IZBORU U ZVANJE

Dr OLEG OBRADOVIĆ bira se u akademsko zvanje redovni profesor Univerziteta Crne Gore za predmete: Parcijalne jednačine na **Prirodno-matematičkom fakultetu** i Matematika I i Matematika II, na nematičnim fakultetima.

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Rodjen sam 18. januara 1964. godine u Gospiću, Hrvatska.

Osnovnu i srednju školu sam završio u Podgorici. Studije fizike na Institutu za matematiku i fiziku sam započeo 1983.godine i završio 1987. godine sa prosječnom ocjenom 9.64. Iste godine sam se zaposlio na Institutu za matematiku i fiziku, današnji Prirodno- matematički fakultet, kao asistent-pripravnik. Poslijediplomske studije sam upisao 1988. godine na Matematičkom fakultetu Univerziteta u Beogradu. Od januara 1989. godine do 1996. godine sam boravio na specijalizaciji na Moskovskom državnom univerzitetu "M.V . Lomonosov" pod naučnim rukovodstvom profesora F.P. Vasiljeva. U januaru 1991. godine odbranio sam magistarski rad pod nazivom: "Regularizacija iterativne splajn- aproksimacije problema optimalnog upravljanja". Te 1991. godine sam izabran u zvanje asistenta na Odsjeku za matematiku.

Kao asistent sam držao vježbe iz predmeta Matematička analiza, Matematičko programiranje i Optimalno upravljanje na Odsjeku za Matematiku, kao i iz predmeta Matematika 1 i Matematika 2 na Elektrotehničkom fakultetu Univerziteta Crne Gore.

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Od izbora u zvanje docenta, odnosno redovnog profesora sam držao predavanja iz predmeta Matematičko programiranje, Parcijalne diferencijalne jednačine na Osjeku za matematiku, Numeričke metode na Odsjeku za fiziku i Mašinskom fakultetu. Dražao sam predavanja na poslijediplomskim studijama iz predmeta Numeričke metode na Mašinskom fakultetu, zatim Matematiku na Odsjeku za biologiju. Posljednjih 6 godina držao sam predavanja iz Parcijalnih diferencijalnih jednačina na Odsjeku za matematiku, kao i predavanja iz predmeta Matematika 1 i Matematika 2 na prvoj godini Elektrotehničkog fakulteta.

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O D L U K U
O IZBORU U ZVANJE

Dr Darko Mitrović bira se u akademsko zvanje **redovni profesor za oblast Matematička analiza i primijenjena matematika na Prirodno-matematičkom fakultetu i na nematičnim fakultetima,** na neodređeno vrijeme.

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2. **Darko Mitrović** je rukovodilac projekta "Advekciono-difuzione jednačine u heterogenim sredinama" finansiranog od strane Ministarstva nauke Crne Gore u periodu 2012-2015.

3. **Darko Mitrović** je rukovodilac bilateralnog projekta "Problemi toka na mnogostrukostima" finansiranog od strane Ministarstva nauke Crne Gore i Ministarstva nauke Austrije u periodu 2015-2017.

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11. Mitrovic, D.: **Existence and Stability of Multidimensional Scalar Conservation Laws with Discontinuous Flux**, *Networks and Heterogeneous Media, Vol.5 (2010), 163-188*

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4. *Mitrovic, D.; Susic, J.:* **Global in time solution to Hopf equation and applications on non-strictly hyperbolic system of conservation laws**, *Electronic Journal of Differential Equations, Vol. 2007(2007), No. 114, 1-22*

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6. *Danilov, V. G.; Mitrovic, D.:* **New approach to shocks generation for conservation laws. Example: global solution to Hopf equation,** *Matematicki Vesnik, 56 (2004), No. 1-2,* 23-46.

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Број 2556
Подгорица, 01.11.2012 год.

Na osnovu člana 75 stav 2 Zakona o visokom obrazovanju (Sl.list RCG, br. 60/03 i Sl.list CG, br. 45/10 i 47/11) i člana 18 stav 1 tačka 3 Statuta Univerziteta Crne Gore, Senat Univerziteta Crne Gore, na sjednici održanoj 25.10.2012. godine, donio je

ODLUKU
O IZBORU U ZVANJE

Dr DAVID KALAJ bira se u akademsko zvanje **redovni profesor** Univerziteta Crne Gore za predmete: Kompleksna analiza 2 (studijski program Matematika), Analiza 3 (studijski program Računarske nauke) i Analiza 3 (studijski program Fizika) na **Prirodno-matematičkom fakultetu.**

REKTOR
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Education

March 2002 University of Belgrade, Faculty of Mathematics
PhD in Mathematics. Thesis title: "Harmonic Mappings and Quasiconformal Harmonic Mappings between Convex Domains".

1995 – 1998 University of Belgrade , Faculty of Mathematics
M.SC, Program. Thesis title: "Harmonic diffeomophisms and quasiconformal mappings"
GPA: 10,00/10.00

1991 - 1995 University of Montenegro, Faculty of Sciences
B.SC in Mathematics
GPA: 9,52/10.00

Fellowships and Awards

1993 Annual fellowship of the Ministry of Education of the Republic of Montenegro

1994 "Decembarska nagrada grada Podgorice" (the Award of the Municipality of Podgorica for distinctive results achieved as a student)

2012 The award for the best project funded by the Ministry of science of Montenegro

Teaching experience

1995 – 1997 Teach. assistant, University of Montenegro, Faculty of Sciences
Mathematical Analysis 2, undergraduate course
Differential Calculus, undergraduate course

1998 – 2002 Teach. assistant, University of Montenegro, Faculty of Sciences
Mathematical Analysis 2, undergraduate course
Complex Analysis, undergraduate course

2002 – 2007 Assist. professor, University of Montegro, Faculty of Sciences
Complex Analysis, undergraduate course
Mathematical Analysis 3, undergraduate course

2007 – 2012 Associate professor, University of Montenegro
Complex Analysis, Mathematics 1, Mathematics 2.

Biografija dr Davida Kalaj

Lični podaci
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Datum i mjesto rođenja: 11. 12. 1971, Podgorica
Institucija
Univerzitet Crne Gore,

- Prirodno-matematički fakultet, UCG

Zvanje: Redovni profesor

- Samostalni studijski program za obrazovanje učitelja na albanskom jeziku, UCG

Funkcija: rukovodilac

A) OBRAZOVANJE
1991: Maturirao Srednju školu „25 Maj", Tuzi, Titograd, sa odličnim uspjehom.
1995: Diplomirao na Prirodno Matematičkom Fakultetu, Univerziteta Crne Gore sa prosječnom ocjenom 9.52. (Za postignuti uspjeh tokom školovanja je 1995. godine dobio Studensku nagradu 19 decembar (Nagrada opštine Podgorice))

1995: Magistrirao na Matematičkom fakultetu, Univerziteta u Beogradu, smjer matematička analiza, sa temom *Harmonijske funkcije i kvazikonformna preslikavanja*, pošto je položio sve predviđene ispite sa ocjenom 10.
2002: Odbranio doktorsku disertaciju pod nazivom *Harmonijske funkcije i kvazikonformne harmonijske funkcije između konveksnih domena* na Matematičkom fakultetu, Univerziteta u Beogradu.

B) NASTAVNA ISKUSTVA

1995 – 1997 | Saradnik u nastavi , Prirodno-Matematičkog fakulteta UCG
1998 – 2002 | Asistent , Prirodno-Matematičkog fakulteta UCG
2002 – 2007 | Docent Prirodno-Matematičkog fakulteta UCG
Kompleksna Analiza, Mathematička Analiza 3 (PMF), Matematike 1,2,3,4
(Studijski program za obrazovanje učitelja na albanskom jeziku)
2007- 2012 | Vanredni professor Prirodno-Matematičkog fakulteta UCG
Kompleksna Analiza, Mathematička Analiza 3 (PMF),
Matematike 1,2,3,4 (Studijski program za obrazovanje učitelja na albanskom jeziku)
Realna i kompleksna analiza (kurs na posdiplomskim studijama PMF-a)
Viša analiza (Kurs na doktorskim studijama PMF i Matematičkog fakulteta u Beogradu)
2012 - | Redovni profesor Prirodno-Matematičkog fakulteta UCG

- Kompleksna Analiza II, Mathematička Analiza 3, Mathematička [illegible] (PMF),
- Matematike 1,2,3,4 (Studijski program za obrazovanje učitelja na [illegible] jeziku)

- Realne i kompleksna analiza (kurs na posdiplomskim studijama PMF)
- Viša analiza (kurs na doktorskim studijama PMF)
- Harmonijske funkcije, doktorski kurs, Prirodno-Matematički fakultet Sarajevo (2014).

Mentorstva na doktorskim disertacijama

2013. Marijan Markovic (Beogradski univerzitet).

2014. Djordjije Vujadinovic (Beogradski univerzitet)

Mentorska na magistarskim tezama

2010. Djordjije Vujadinovic (UCG)

C). NAUČNO-ISTRAŽIVAČKI INTERES

Geometrijska teorija funkcija: Harmonijske funkcije, Kvazikonformna preslikavanja, Holomorne funkcije. Funkcijonalni prostori: Hardijevi i Bergmanovi prostori, Parcijalne diferencijalne jednačine: Poissonova, Laplaceova, Eliptičke PDE, Diferencijalna geometrija: Harmonijske površi, Minimalne površi, Izoperimetrija nejednakost itd.

- Upravljanje projektima

1. Rukovodilac nacijonalnog projekta Analiza na mnogostrukosti i primjene (2011-2015), koga finansira Ministarstvo nauke Republike Crne Gore. Projektni tim čine renomirani matematičari iz Crne Gore. Pri tome je projekat pri evaluaciji osvojio maksimalan broj poena od strane međunarodnih ekperata. (Nagrada Ministarstva nauke za najbolji naučni projekat za 2013 godinu)

3. Trenutno je rukovodilac dva biralateralna projekta jednog sa Kinom i drugog sa Hrvatskom.

2. Bio je Rukovodilac uspješnog nacijonalnog projekta Analiza na mnogostrukosti (2008-2011).

- Izvod iz bibliografije

Publikovani (ukupno 65 radova), između ostalog, u sledećim vrhunskim matematičkim časopisima: Advances in Mathematics, Transactions of American Mathematical Society, Calculus of Variations and PDE, International mathematics research noticies, Poceedings of American Mathematical Society, Journal D' Analyse Math. Israel Journal of Math, Mathematische Zeitschrift, Annali della Scuola Normale Superiore di Pisa- Classe di Scienze, Annales Academiæ Scientiarium Fennicæ Mathematica, Annali di Matematica Pura ed Applicata, Pacific Journal of Mathematics.

U pripremi ima još 6 radova koji se nalaze na arxiv.org serveru.

Ukupan broj radova publikovanih na žurnalima koji pripadaju SCI listi je 60. Svoje radove je izložio na više od 25 naučnih konferencija i seminara i to u sljedeće države: SAD, Rusija, Japan, Kina, Južna Koreja, Njemačka, Francuska, Finska, Rumunija, Srbija, Norveška, Češka, Poljska itd. Njegovi koautori su između ostalog: Noam Elkies, Eero Saksman, Matti Vuorinen, Miroslav Pavlović, Miodrag Mateljević koji su dali svojevrstan pečat modernoj matematici. Kao dokaz ove teze je činjenica da je Noam Elkies svojevremeno postao najmlađi profesor u historiji Harvard univerziteta (http://en.wikipedia.org/wiki/Noam_Elkies), dok je Eero Skaksman urednik Acta Mathematica"), koji je najprestižniji svjetski matematički časopis. (http://www.springer.com/mathematics/journal/11511?detailsPage=editorialBoard Kalajevi radovi su citirani više od 550 puta (www.googlescholar.com). (Spisak radova i konferencija su dati u prilogu).

D) Udžbenici

1. D. Kalaj: Zbirka zadataka iz kompleksne analize, Univerzitet Crne Gore, 2006, 219 str.
2. M. Jaćimović, D. Kalaj: Uvod u kompleksnu analizu, Univerzitet Crne Gore, 2009, 347 str.

Prevodi i adaptacija školskih udžbenika

Prevod i adaptacija udžbenika iz matematike za ukupno 8 razreda za osnovnu i srednju školu sa srpskog (crnogorskog) na albanski jezik u izdanju izdavačke kuće "Zavod za užbenike i nastavna sredstva" u periodu 2008-2010 i 2014.

E) Uredništva

Urednik je sljedećih matematičkih časopisa:
1. World scientific journal
(http://www.hindawi.com/journals/tswj/editors/mathematical.analysis/)
2. Abstract and applied mathematics,
http://www.hindawi.com/journals/aaa/psi/
3. Bulletin of mathematical analysis and application,
http://91.187.98.171/bmathaa/
4. Albanian journal of mathematics

F) Recenzije i ekspertize:

Recenzija radova za renomirane časopise:
Transaction of AMS, Indiana Journal of mathematics, Proceeding of AMS, Annales Academiæ Scientiarium Fennicæ Mathematica, Applied Mathematics Letters, Abstract and Applied Analysis, Applied Mathematics and Computation, Complex Variables and Elliptic equations, Filomat, Publications d'Institute mathematique, Bulletin of the Malaysian Mathematical Sciences Society, Journal of mathematical analysis and application, Bulletin of mathematical analysis and application, Publicationes Mathematicae Debrecen, Worlds scientific journal, Acta Mathematica

Sinica, Turkish journal of math, Mathematica slovaca, Bulletin of London math society, Journal of Indian Academy of Mathematics etc.

Međunarodne ekspertize za projekte:

- Evaluator za projekte iz oblasti matematika koje je raspisalo Israel science fondation države Israel na period 2009-2012.
- - Evaluator za projekte iz oblasti matematika koje je raspisalo Israel science fondation države Israel na period 2012-2015.
- - Evaluator za projekte iz oblasti matematika koje je raspisalo Israel science fondation države Israel na period 2015-2018.
- - Evaluator za projekte iz oblasti matematika koje je raspisalo Ministarstvo prosvjete i nauke Republike Srbije na period 2011-2014.
- - Evaluator za projekte FONDECYT, Chile 2014.

Mathematics 3, Mathematics 4, (Study programme [illegible] teachers in Albanian Language) undergraduate course.
Mathamatical Analysis 3, undergraduate course
Real and Complex Analysis, graduate course

2012 – Full professor, University of Montenegro

Mentorstva na doktorskim disertacijama
2013, Marijan Markovic (Beogradski univerzitet)
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2010, Djordjije Vujadinovic

DODATNE INFORMACIJE

Born: December 11, 1971, Podgorica, Yugoslavia
Citizenship: Montenegrin
Languages: Albanian (native command), Serbian (native command), English (fluent), Russian (passive), Italian (passive).
Computer skills: Latex, C++, Mathematica software

Projects: a) Establishment and management of Study programme for teacher education at Albanian since 2004.
b) PI of the national project Analysis on manifolds (2008-2011).
c) PI of the national project Analysis on manifolds and applications (2012-2015)

RADOVI

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2. D. Kalaj, *On the Nitsche's conjecture for harmonic mappings* Mathematica Montisnigri Vol XIV (2001) 89-94.
3. D. Kalaj, *The Jacobian of harmonic function and of its boundary values*, Revue Roumaine De Mathematiques Pures Et Appliquees Tome XLVII, N.5-6, (2002).
4. D. Kalaj, *On harmonic diffeomorphisms of the unit disc onto a convex domain*, Complex Var. Theory Appl. 48, No.2, 175-187 (2003).
5. D. Kalaj, *Quasiconformal harmonic functions between convex domains*, Publ. Inst. Math., Nouv. Ser. 76(90), 3-20 (2004).
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7. D. Kalaj, M. Pavlovic, *Boundary correspondence under harmonic quasiconformal homeomorfisms of a half-plane* Ann. Acad. Sci. Fenn. Math. 30 (2005), no. 1, 159-165.
8. D. Kalaj, *On the Nitsche conjecture for harmonic mappings in R^2 and in R^3*. Israel J. Math. 150 (2005) 241-251.
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10. Sh. Najafzadeh, S. R. Kulkarni and D. Kalaj *Application of convolution and [illegible] operators on analytic and p-valent functions*, Filomat 20:2 (2006) 115–124.
11. D. Kalaj [illegible] *of PDE's Delta u=0 between spherical* [illegible] Math. Anal. Appl. Volume 327, Issue 1, Pages 1-11 (2007).
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13. D. Kalaj, M Mateljević, *Quasiconformal and harmonic mappings between [illegible] Jordan domains*, Novi Sad J. of Mathematics, 38 (2) 2008, 147-156.
14. D. Kalaj, *On harmonic quasiconformal self-mappings of the unit ball*, Ann. Acad. Sc. Fenn. Math. Vol 33, 261-271, (2008).
15. D. Kalaj, *Lipschitz spaces and harmonic mappings*, Ann. Acad. Sci. Fenn. Math. Vol 34, 2009, 475-485.
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17. D. Kalaj, M Mateljevic, *Harmonic q.c. self-mapping and Möbius transformations of the unit ball B^n*, Pacific J. Math. Vol. 247, No. 2, 2010, 389–406.
18. D. Kalaj, *On an integral inequality and application to Poission equation*, Applied Mathematics Letters, 23 (2010) 1016–1020.
19. D. Kalaj *Quasiconformal harmonic mappings and close to convex domains*, Filomat, Volume 24, Number 1, April 2010, 63—68.
20. D. Kalaj, M Mateljevic, *On absolutely conformal mappings*, Publ. Math. Debrecen. 77/1-2 (2010), 33-38.
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24. D. Kalaj, *Harmonic maps between annuli on Riemann surfaces*, Israel J. Math. 182 (2011), 123—147.
25. D. Kalaj, M Mateljevic, *On quasiconformal harmonic surfaces with rectifiable boundary*, Complex Complex Anal. Oper. Theory 5, No. 3, 633-646 (2011).
26. D. Kalaj, M Mateljevic, *On certain nonlinear elliptic PDE and quasiconformal maps between Euclidean surfaces*, Potential analysis, Volume 34, Number 1, 13-22, DOI: 10.1007/s11118-010-9177-x (10 pages).
27. D. Kalaj, *Harmonic mappings and distance function* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5), Vol. X (2011), 669-681.
28. D. Kalaj, *Isoperimetric inequality for the polydisk*, Annali di matematica pura ed applicata, Volume 190 (2011), Number 2, 355-369.
29. D. Kalaj, *Estimates of gradient and of Jacobian of harmonic mappings defined in the unit disk*. Proc. Am. Math. Soc. 139, No. 7, 2463-2472 (2011).
30. D. Kalaj, *Invertible harmonic mappings beyond Kneser theorem and quasiconformal harmonic mappings*, Stud. Math. 207, No. 2, 117-136 (2011), arXiv:1003.2740.
31. D. Kalaj, R. Meštrović, *An isoperimetric type inequality for harmonic functions*, Journal of Mathematical Analysis and Applications, Volume 373, Issue 2, 15 January 2011, Pages 439-448.
32. D. Kalaj, *On the quasiconformal self-mappings of the unit ball satisfying the Poisson differential equations*, Ann. Acad. Sci. Fenn. Math. Volumen 36, 2011, 177-194.
33. D. Kalaj, M Mateljevic, *(K,K')-quasiconformal harmonic mappings*, Potential Anal. 36, No. 1, 117-135 (2012).
34. *D. Kalaj, M. Vuorinen: On harmonic functions and the Schwarz lemma*. Proc. Amer. Math. Soc. 140 (2012), 161-165.
35. Barbara A. Bhayo, Vladimir Božin, David Kalaj, Matti Vuorinen: *[illegible]*, Journal of Mathematical Analysis and Applications [illegible] Issue [illegible] August 2015, Pages 768-781.
36. *D. Kalaj, M. Mateljevic: On conformal, harmonic mappings and Dirichlet's integral*, Filomat Volume 25, Number 2, June 2011, 91-108
37. *D. Kalaj, On boundary correspondence of q.c. harmonic mappings between [illegible] domains*, Math. Nachr. 285, No. 2-3, 283-294 (2012).
38. *D. Kalaj, M. [illegible]: Optimal estimates for the harmonic functions* [illegible] 771-782 (2012).

39. D. Kalaj, *Quasiconformal harmonic mappings between $C^{1,\alpha}$ Euclidean surfaces*. Monatsh. Math. 167, No. 2, 205-229 (2012).

40. *D. Kalaj, V. Manojlovic: Subharmonicity of the modulus of quasiregular harmonic mappings*. Journal of mathematical analysis and applications Volume 379, Issue 2, 15 July 2011, Pages 783-787.

41. D. Kalaj, *On Kellogg's theorem for quasiconformal mappings*. Glasg. Math. J. 54, No. 3, 599-603 (2012).

42. D.. Kalaj, *On some integrable operators related to Poisson equation*, Integral Equations Oper. Theory 72, No. 4, 563-575 (2012).

43. D. Kalaj, *Stolz angle limit of a certain class of selfmappings of the unit disk*, J. Approx. Theory 164, No. 6, 815-822 (2012).

44. D. Kalaj, *Cauchy transform and Poisson's equation*, Advances in Mathematics 231, No. 1, 213-242 (2012).

45. *D. Kalaj, On quasiconformal selfmappings of the unit disk and elliptic PDE in the plane. Proceedings of the Royal Society of Edinburgh: Section A, Volume 143 / Issue 04 / August 2013 pp 831-849.*

46. *D. Kalaj, M. Markovic: Optimal estimates for the gradient of harmonic functions in the unit disk. Complex Anal. Oper. Theory 7, No. 4, 1167-1183 (2013).*

47. D. Kalaj, *A priori estimate of gradient of a solution to certain differential inequality and quasiregular mappings*, J. Anal. Math. 119, 63-88 (2013).

48. D. Kalaj, S. Ponnusamy, M. Vuorinen, *Radius of Close-to-convexity of Harmonic Functions*, Complex Var. Elliptic Equ. 59, No. 4, 539-552 (2014).

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KNJIGE

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PREVODI

Prevod i adaptacija sljedećih udžbenika iz matematike sa srpskog (crnogorskog) na albanski jezik u izdanju izdavačke kuće "Zavod za udžbenike i nastavna sredstva" u periodu 2008-2010.

1) Matematika IV (četvrti razred osnovne škole)
2) Matematika V (peti razred osnovne škole)
3) Matematika VIII (osmi razred osnovne škole)
4) Matematika IX (deveti razred osnovne škole)
5) Matematika II (drugi razred srednje škole)
6) Matematika III (treći razred srednje škole).
7) Algoritmi i programiranje (treći i četvrti razred srednje škole).

Citati: 550 citata (http://scholar.google.com).

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Na osnovu člana 72 stav 2 Zakona o visokom obrazovanju („Službeni list Crne Gore" br. 44/14, 47/15,40/16,42/17,71/17) i člana 32 stav 1 tačka 9 Statuta Univerziteta Crne Gore, Senat Univerziteta Crne Gore, na sjednici održanoj 09.10. 2018.godine, donio je

O D L U K U
O IZBORU U ZVANJE

Dr GORAN POPIVODA bira se u akademsko zvanje **docent Univerziteta Crne Gore za oblast Vjerovatnoća i statistika sa primjenama** na Prirodno-matematičkom fakultetu Univerziteta Crne Gore i na nematičnim fakultetima, na period od pet godina.

SENAT UNIVERZITETA CRNE GORE
PREDSJEDNIK

UNIVERZITET CRNE GORE UCG 1974

Prof.dr Danilo Nikolić, rektor

Goran Popivoda

Biografija

Goran Popivoda je rođen na Cetinju, 9. oktobra 1984. godine. Osnovnu školu i gimnaziju završio je u rodnom gradu. Dobitnik je diplome „Luča" za odličan uspjeh u svim razredima osnovnog i srednjeg školovanja. Četvorogodišnje studije na Prirodno-matematičkom fakultetu u Podgorici, smjer Matematika i računarske nauke, završio je 2007. godine, sa prosječnom ocjenom 9,76.

U toku studiranja bio je dobitnik stipendije koju Vlada Republike Crne Gore dodjeljuje talentovanim učenicima i studentima i slične stipendije Opštine Cetinje.

Magistarski rad pod nazivom „Vinerov proces", odbranio je u septembru 2010. godine. U martu 2011. godine upisuje doktorske studije na Prirodno-matematičkom fakultetu na smjeru Matematika, a doktorsku disertaciju pod nazivom „Ekstremi uslovno-Gausovih procesa" odbranio je 28. oktobra 2017. godine. Na magistarskim i doktorskim studijama radio je pod rukovodstvom prof. dr Siniše Stamatovića.

Jedan je od koordinatora takmičenja Olimpijada znanja i član je Komisije na Državnom takmičenju iz matematike od 2008. godine. Na Balkanskoj matematičkoj olimpijadi (od 2011. godine do 2019. godine) i Međunarodnoj matematičkoj olimpijadi (od 2018. godine) je vođa tima.

Na Prirodno-matematičkom fakultetu, Univerziteta Crne Gore, od februara 2008. do oktobra 2018. radio je kao saradnik u nastavi. Izvodio je vježbe na predmetima: Teorija vjerovatnoće, Vjerovatnoća i statistika, Statistika, Analiza 1, Analiza 2, Uvod u kombinatoriku, Diskretna matematika, Diskretna matematika 1, Diskretna matematika 2, Slučajni procesi, Metode optimizacije, Lanci Markova, Matematika V, Aktuarska matematika, Osnovne matematičke i statističke metode, Matematika, Matematika 1 i Matematika 2 (posljednja četiri predmeta na Metalurško-tehnološkom fakultetu).

U oktobru 2018. godine izabran je u zvanje docenta Univerziteta Crne Gore. Izvodi nastavu na predmetima: Vjerovatnoća i statistika, Lanci Markova, Istorija i filozofija matematike, Teorija izračunljivosti, Statistika u farmaciji (na Medicinskom fakultetu) i Matematika IV (na Građevinskom fakultetu).

Oblasti njegovog naučnog interesovanja su: teorija vjerovatnoće, slučajni procesi, ekstremi Gausovih procesa, statistika i diskretna matematika.

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УНИВЕРЗИТЕТ У НОВОМ САДУ
ПРИРОДНО-МАТЕМАТИЧКИ ФАКУЛТЕТ
ПРИМЉЕНО 31-01-2020
ОРГАНИЗ.ЈЕД. 0601 БРОЈ 692/14

Број: 04-29/17
Нови Сад, 30. јануар 2020. године

На основу члана 58 став 3 тачка 5 и члана 75 Закона о високом образовању („Службени гласник РС" број 88/2017, 27/2018 – др. закон, 73/2018 и 67/2019), члана 67 став 1 тачка 5 Статута Универзитета у Новом Саду број 01-173/1 од 13. фебруара 2019. године, чланова 2, 3 и 4 Правилника о ближим минималним условима за избор у звање наставника на Универзитету у Новом Саду број 04-179/10 од 9. октобра 2018. године и члана 7 Правилника о начину и поступку стицања звања и заснивања радног односа наставника Универзитета у Новом Саду број 04-179/7 од 12. јула 2018. године, Сенат Универзитета у Новом Саду на седници одржаној 30. јануара 2020. године, једногласно је донео

О Д Л У К У

Др Сања Коњик бира се у звање редовног професора за ужу научну област Анализа и вероватноћа на Природно-математичком факултету Универзитета у Новом Саду.

Одлука се примењује од дана закључења уговора о раду лица изабраног у звање наставника из става 1 ове одлуке са деканом Факултета.

О б р а з л о ж е њ е

На основу одлуке декана Природно-математичког факултета Универзитета у Новом Саду објављен је конкурс за избор наставника у звање ванредног или редовног професора за ужу научну област Анализа и вероватноћа на Департману за математику и информатику Природно-математичког факултета Универзитета у Новом Саду. Конкурс је објављен у листу Дневник дана 27. септембра 2019. године.

На објављени конкурс пријавила се кандидаткиња: др Сања Коњик.

Одлуком Изборног већа Департмана за математику и информатику Природно-математичког факултета Универзитета у Новом Саду број 0601-692/7 од 9. октобра 2019. године именована је Комисија за писање реферата о пријављеним кандидатима за избор у звање наставника, у следећем саставу:

- Др Стеван Пилиповић, редовни професор Природно-математичког факултета Универзитета у Новом Саду (ужа научна област Анализа и вероватноћа)
- Др Марко Недељков, редовни професор Природно-математичког факултета Универзитета у Новом Саду (ужа научна област Анализа и вероватноћа)
- Др Данијела Рајтер-Ћирић, редовни професор Природно-математичког факултета Универзитета у Новом Саду (ужа научна област Анализа и вероватноћа)
- Др Татјана Дошеновић, редовни професор Технолошког факултета Нови Сад Универзитета у Новом Саду (ужа научна област Анализа и вероватноћа)

Комисија за писање реферата о кандидатима за избор у звање наставника је дана 29. октобра 2019. године доставила Изборном већу Департмана за математику и информатику Природно-математичког факултета Универзитета у Новом Саду, реферат број 0601-692/8 од 29. октобра

2019. године у коме је утврдила предлог да се др Сања Коњик изабере у звање редовног професора.

Реферат Комисије стављен је на увид јавности 1. новембра 2019. године, објављивањем на интернет страници Универзитета у Новом Саду, у Билтену бр. 1592 од 1. новембра 2019. године.

Изборно веће Департмана за математику и информатику Природно-математичког факултета Универзитета у Новом Саду на седници одржаној 9. децембра 2019. године утврдило је резултате рада:

1. обавезни елементи:

1.1. Способност за наставни рад или резултати у наставном раду у претходном изборном периоду

1.2. Способност за научно-истраживачки, односно уметнички рад или резултати у научно-истраживачком, односно уметничком раду у претходном изборном периоду

2. изборни елементи:

2.1. Стручно-професионални допринос

2.2. Допринос академској и широј заједници

2.3. Сарадња са другим високошколским, научно-истраживачким, односно институцијама културе или уметности у земљи и иностранству

и утврдило Предлог одлуке о избору др Сање Коњик у звање редовног професора.

Природно-математички факултет Универзитета у Новом Саду доставио је документацију прописану чланом 4 Правилника о начину и поступку стицања звања и заснивања радног односа наставника Универзитета у Новом Саду Стручном већу за природно-математичке науке Сената Универзитета у Новом Саду.

Стручно веће за природно-математичке науке Сената Универзитета у Новом Саду на седници одржаној дана 17. јануара 2020. године дало је позитивно мишљење о предлогу одлуке о избору др Сање Коњик у звање редовног професора.

Имајући у виду сву достављену документацију, Сенат Универзитета је на седници одржаној 30. јануара 2020. године једногласно донео одлуку да се др Сања Коњик изабере у звање редовног професора за ужу научну област Анализа и вероватноћа на Природно-математичком факултету Универзитета у Новом Саду.

ПОУКА О ПРАВНОМ ЛЕКУ:

Ова одлука је коначна и против ње незадовољни учесници Конкурса могу покренути управни спор пред Управним судом у Београду, Немањина 9, у року од 30 дана од дана пријема. За подношење тужбе за покретање управног спора предвиђена је такса у износу од 390 динара.

Проф. др Дејан Јакшић
Председник Сената Универзитета

Одлуку доставити:
1. Лицу изабраном у звање наставника путем Факултета
2. Природно-математичком факултету Универзитета у Новом Саду
3. Архиви Универзитета у Новом Саду

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EDUCATION

1995-1999	Undergraduate studies in Mathematics, Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, Serbia
1999	BSc in Mathematics
1999-2003	Postgraduate studies in Mathematics, Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, Serbia
2003	MSc in Mathematics Master Thesis "Symmetry Groups of Systems of Conservation Laws", Advisor: Academician Prof. dr Stevan Pilipović
2004-2008	Doctoral studies in Mathematics, Faculty of Mathematics, University of Vienna, Austria
2008	PhD in Mathematics PhD Thesis "Group Analysis and Variational Symmetries for Non-smooth Problems", Advisor: Ao. Univ.-Prof. Dr. Michael Kunzinger

ACADEMIC EMPLOYMENT

1999-2004	Junior Teaching Assistant, Faculty of Agriculture, University of Novi Sad, Serbia
2004-2009	Teaching Assistant, Faculty of Agriculture, University of Novi Sad, Serbia
2009-2010	Assistant Professor, Faculty of Agriculture, University of Novi Sad, Serbia
2010-2015	Assistant Professor, Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, Serbia
2015-2020	Associate Professor, Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, Serbia
Since 2018	Visiting Professor, Faculty of Applied Sciences, University of Donja Gorica, Montenegro
Since 2020	Full Professor, Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, Serbia

ACADEMIC SERVICE

Since 2015	Head of Applied Analysis Group at Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, Serbia
Since 2015	Member of the Board of the Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, Serbia
Since 2015	Head of the Ethics Commission of the Faculty of Sciences, University of Novi Sad, Serbia

LANGUAGES

Mother tongue: Serbian
Active knowledge: English
Reading knowledge: German, other Slavic languages

AWARDS, GRANTS

1996	University of Novi Sad Award for achievements during the studies
1997	University of Novi Sad Exceptional Award for achievements during the studies
1998	University of Novi Sad Exceptional Award for achievements during the studies
1999	University of Novi Sad Exceptional Award for achievements during the studies
2003	One-month ÖAD Research Grant
2003-2004	8-Month ÖAD Research Grant

MEMBERSHIPS IN SCIENTIFIC AND PROFESSIONAL ORGANIZATIONS

ISAAC	The International Society for Analysis, its Applications and Computation (Life member)
IAGF	The International Association for Generalized Functions (Executive Editor)
EWM	European Women in Mathematics
SIAM	Society for Industrial and Applied Mathematics
SMSA	Serbian Mathematical Sciences Association

SELECTED RESEARCH PROJECTS

member	START Project Y 237 Nonlinear Distributional Geometry Project leader: Michael Kunzinger Duration: 2005-2011
leader	APV 114-451-2167/2011-12 Methods of Functional Analysis and Fractional Calculus with Applications in Mechanics Project leader: Sanja Konjik Duration: 2011-2015
member	MNTR 174005 Viscoelasticity of Fractional Type and Shape Optimization in a Theory of Rods Project leader: Academician Teodor M. Atanacković Duration: 2011-2022
member	MNTR 174024 Methods of Functional and Harmonic Analysis and PDE with Singularities Project leader: Academician Stevan Pilipović Duration: 2011-2022
member	TEMPUS JP 511140-2010 Master programme in applied statistics Project leader: Nevenka Žarkić-Joksimović, Miroslav Vesković Duration: 2010-2013
member	TEMPUS 530550-2012 Towards Sustainable and Equitable Financing of Higher Education in Bosnia and Herzegovina, Montenegro and Serbia Project leader: Andreja Tepavčević Duration: 2012-2015
leader	APV 114-451-2098 Analytical, numerical and statistical tools in mathematical models Project leader: Sanja Konjik Duration: 2016-2020
MC member	Cost Action CA15225 FRACTAL Fractional-order systems; analysis, synthesis and their importance for future design MC Chair: Jaroslav Koton, Brno University of Technology Duration: 2016-2020
leader	Bilateral Project with Montenergo - Fractional and cellular automata models of wave propagation: Analysis, sinthesis and application Project leaders: Biljana Stamatović (Montenegro) & Sanja Konjik (Serbia) Duration: 2017-2018
member	Bilateral Project with Austria - Functional analytic methods for models of wave propagation in viscoelastic media Project leaders: Günther Hörmann (Austria) & Ljubica Oparnica (Serbia) Duration: 2018-2019
leader	Bilateral Project with Croatia – Applied mathematical analysis tools in modeling biophysical phenomena Project leaders: Davor Horvatić (Croatia) & Sanja Konjik (Serbia) Duration: 2019-2020

SELECTED RESEARCH STAYS, CONFERENCES, SCHOOLS

23-28 Oct 2000	Perturbative Methods for Partial Differential Equations and Dynamical Systems, Cagliari, Italy
24 Sep-12 Oct 2001	Intensive Course "Wavelet Analysis and Signal Processing", Sofia, Bulgaria
Oct 2003-May 2004	University of Vienna, Institute of Mathematics
22-28 Sep 2004	International Conference "Generalized Functions 2004; Topics in PDE, Harmonic Analysis and Mathematical Physics", Novi Sad, Serbia
Oct 2004-Jan 2005	University of Vienna, Faculty of Mathematics
Feb-June 2006	University of Vienna, Faculty of Mathematics
May-June 2007	University of Vienna, Faculty of Mathematics
2-8 Sep 2007	International Conference "Linear and Nonlinear Theory of Generalized Functions and its Applications", Będlewo, Poland
13-18 July 2009	7th International ISAAC Congress, London, UK
31 Aug-4 Sep 2009	International Conference on Generalized Functions GF2009, Vienna, Austria
18-20 Oct 2010	FDA'10 - 4th IFAC Workshop on Fractional Differentiation and Its Applications, Badajoz, Spain
18-22 Apr 2011	International Conference GF2011 on Generalized Functions, Linear and Nonlinear problems, Martinique, France
22-27 Aug 2011	8th International ISAAC Congress, Moscow, Russia
14-17 May 2012	FDA'12 - 5th Symposium on Fractional Differentiation and Its Applications, Nanjing, China
18-22 Mar 2013	GAMM 2013 - 84th Annual Meeting of the International Association of Applied Mathematics and Mechanics, Novi Sad, Serbia
10-12 July 2013	International Conference "Generalized Functions and Nonlinear Problems", Campinas, Brazil
5-9 Aug 2013	9th International ISAAC Congress, Krakow, Poland
2-6 Sep 2013	16th General Meeting of EWM, Bonn, Germany
23-25 Jun 2014	ICFDA'14 - 2014 International Conference on Fractional Differentiation and Its Applications, Catania, Italy
12-14 Aug 2014	ICWM 2014 – International Congress of Women Mathematicians, Seoul, Korea
13-21 Aug 2014	SEOUL ICM 2014 – International Congress of Mathematicians, Coex, Seoul, Korea
8-12 Sep 2014	International Conference on Generalized Functions GF2014, Southampton, UK
10-13 Jun 2015	AMS-EMS-SPM International Meeting 2015, Porto, Portugal
3-8 Aug 2015	10th International ISAAC Congress, Macau, China
31 Aug-4 Sep 2015	17th General Meeting of EWM, Cortona, Italy
4-9 Sep 2016	International Conference on Generalized Functions GF2016, Dubrovnik, Croatia
26-30 Sep 2016	International Conference "Analysis and Partial Differential Equations", London, UK
14-18 Aug 2017	11th International ISAAC Congress, Växjö, Sweden
9-12 Dec 2017	SIAM Conference on Analysis of Partial Differential Equations, Baltimore, Maryland, U.S.
10-11 May 2018	Workshop on Fractional Calculus, Skopje, Republic of Macedonia
1-9 Aug 2018	ICM 2018 – International Congress of Mathematicians, Rio de Janeiro, Brazil
27-31 Aug 2018	International Conference on Generalized Functions GF2018, Novi Sad, Serbia
3-7 Sep 2018	18th General Meeting of EWM, Graz, Austria
25-28 Jun 2019	Barcelona Analysis Conference 2019, Barcelona, Spain
29 Jul-2 Aug 2019	12th International ISAAC Congress, Aveiro, Portugal
11-14 Dec 2019	SIAM Conference on Analysis of Partial Differential Equations, La Quinta, California, U.S.
16-30 Jan 2020	University of Ghent, Department of Mathematics: Analysis, Logic and Discrete Mathematics

CONFERENCES AND SCHOOLS ORGANIZATION

2007	Member of the Organizing Committee of the International Conference "MM-VII Symmetries and Mechanics", Novi Sad, Serbia, 2007.
2008	Member of the Organizing Committee of the 12th Serbian Mathematical Congress, Novi Sad, Serbia, 2008.
2008	Member of the Organizing Committee of the 3rd Serbian-Greek Symposium "Recent Advances in Mechanics", Novi Sad, Serbia, 2008.
2010	Member of the Organizing Committee of the Symposium - Summer School "Generalized Functions in PDE, Geometry, Stochastics and Microlocal Analysis", Novi Sad, Serbia, 2010.

2012 Member of the Organizing Committee of the International Conference "Topics in PDE, Microlocal and Time-frequency Analysis", Novi Sad, Serbia, 2012.
2015 Member of the Scientific Committee of the International Conference "The Cape Verde International Days on Mathematics 2015", Mindelo, São Vicente, Cape Verde, 2015.
2015 Member of the Organizing Committee of the Symposium "Mechanics through Mathematical Modelling", Novi Sad, Serbia, 2015.
2016 Member of the Scientific and Organizing Committee of the International Conference "Applications of Generalized Functions in General Relativity, Stochastics and Mechanics", Novi Sad, Serbia, 2016.
2017 Member of the Scientific Committee of the International Conference "The Cape Verde International Days on Mathematics 2017", Praia, Cape Verde, 2017.
2017 Member of the Scientific Committee of the International Conference "Applications of Generalized Functions in Harmonic Analysis, Mechanics, Stochastics and PDE", Novi Sad, Serbia, 2017.
2018 Member of the Organizing Committee of the International Conference on Generalized Functions GF2018, Novi Sad, Serbia, 2018.

PUBLICATIONS

33. Djordjevic, J., Konjik, S., Mitrovic, D., Novak, A., Global controllability for quasilinear non-negative definite system of ODEs and SDEs, J. Optimization Theory Appl., 190, 316-338, 2021.
32. Goles, N., Nerancic, M., Konjik, S., Pajic-Eggspuehler, B., Pajic, B., Cvejic, Z., Phacoemulsification and IOL-Implantation without using viscoelastics: combined modelling of thermo fluid dynamics, clinical outcomes, and endothelial cell density, Sensors, 21(7), 2399, 2021., https://doi.org/10.3390/s21072399
31. Atanacković, T. M., Konjik, S., Pilipović, S., Variational problems of Herglotz type with complex order fractional derivatives and less regular Lagrangian, Acta Mech., 230, 4357-4365, 2019.
30. Konjik, S., Oparnica, Lj., Zorica, D., Distributed order fractional constitutive stress-strain relation in wave propagation modeling, *Z. Angew. Math. Phys.*, 70:51, 10.1007/s00033-019-1097-z, 2019.
29. Atanacković, T. M., Konjik, S., Pilipović, S., *Wave equation involving fractional derivatives of real and complex fractional order, In A. Kochubei, Y. Luchko (Eds.), Handbook of Fractional Calculus with Applications, Volume 2 Fractional Differential Equations, De Gruyter, 327–352, 2019.* https://doi.org/10.1515/9783110571660-015
28. Atanacković, T. M., Konjik, S., Pilipović, S., Variational principles with fractional derivatives, In A. Kochubei, Y. Luchko (Eds.), Handbook of Fractional Calculus with Applications, Volume 1 Basic Theory, De Gruyter, 361–384, 2019. https://doi.org/10.1515/9783110571622-015
27. Atanacković, T. M., Janev, M., Konjik, S., Pilipović, S., Complex fractional Zener model of wave propagation in $\R$, *Frac. Calc.* Appl. Anal., 21(5), 1313-1334, 2018.
26. Atanacković, T. M., Janev, M., Konjik, S., Pilipović, S., Wave equation for generalized Zener model containing complex order fractional derivatives, *Contin. Mech. Thermodyn.*, 29(2), 569-583, 2017.
25. Atanacković, T. M., Konjik, S., Pilipović, S., Zorica, D., Complex order fractional derivatives in viscoelasticity, *Mech. Time-Depend. Mater.*, 20(2), 175-195, 2016.
24. Atanacković, T. M., Janev, M., Konjik, S., Pilipović, S., Zorica, D., Vibrations of an elastic rod on a viscoelastic foundation of complex fractional Kelvin-Voigt type, *Meccanica*, 50(7), 1679-1692, 2015.
23. Hörmann, G., Konjik, S., Kunzinger, M., A regularization approach to non-smooth symplectic geometry, *In S. Pilipović, J. Toft (Eds.), Pseudo-Differential Operators and Generalized Functions, Oper. Theory, Adv. Appl.*, Birkhäuser/Springer, 245, 117-130, 2015.
22. Hörmann, G., Konjik, S., Kunzinger, M., Symplectic modules over Colombeau-generalized numbers, *Commun. Algebra*, 42, 3558-3577, 2014.
21. Atanacković, T. M., Janev, M., Konjik, S., Pilipović, S., Zorica, D., Expansion formula for fractional derivatives in variational problems, *J. Math. Anal. Appl.*, 409(2), 911-924, 2014.
20. Hörmann, G., Konjik, S., Oparnica, Lj., Generalized solutions for the Euler-Bernoulli model with Zener viscoelastic foundations and distributional forces, *Anal. Appl.*, 11(2), 1350017 (21 pages), 2013.
19. Atanacković, T. M., Konjik, S., Oparnica, Lj., Zorica, D., The Cattaneo type space-time fractional heat conduction equation, *Contin. Mech. Thermodyn.*, 24(4-6), 293-311, 2012.
18. Konjik, S., Atanacković, T. M., Oparnica, Lj., Zorica, D., A note on the constitutive equation in a linear fractional viscoelastic body model, *In V. I. Burenkov, M. L. Goldman, E. B. Laneev, V. D. Stepanov (Eds.), Progress in Analysis. Proceedings of the 8th Congress of the International Society for Analysis, its Applications, and Computation. Moscow, Russia (22-27 August 2011)*, Volume 1, 274-281, 2012.
17. Atanacković, T. M., Konjik, S., Pilipović, S., Fractionalization of constitutive equations in viscoelasticity, *In W. Chen, H. G. Sun, D. Baleanu (Eds.), Proceedings of FDA'12. The 5th Symposium on Fractional Differentiation and its Applications. Nanjing, China (14-17 May 2012)*, Article no. FDA12-129, 6 pages, 2012.

16. Atanacković, T. M., Konjik, S., Oparnica, Lj., Zorica, D., Thermodynamical restrictions and wave propagation for a class of fractional order viscoelastic rods, *Abstr. Appl. Anal.*, 2011(ID 975694, 32 pages), 2011.
15. Konjik, S., Oparnica, Lj., Zorica, D., Waves in viscoelastic media described by a linear fractional model, *Integral Transforms Spec. Funct.*, 22(4-5), 283-291, 2011.
14. Atanacković, T. M., Dolićanin, D., Konjik, S., Pilipović, S., Dissipativity and stability for a nonlinear differential equation with distributed order symmetrized fractional derivative, *Appl. Math. Lett.*, 24(6), 1020-1025, 2011.
13. Konjik, S., Oparnica, Lj., Zorica, D., Waves in fractional Zener type viscoelastic media, *J. Math. Anal. Appl.*, 365(1), 259-268, 2010.
12. Atanacković, T. M., Konjik, S., Oparnica, Lj., Pilipović, S., Generalized Hamilton's principle with fractional derivatives, *J. Phys. A, Math. Theor.*, 43, 255203(12pp), 2010.
11. Atanacković, T. M., Konjik, S., Oparnica, Lj., Pilipović, S., Simić, S., Recent progress in the calculus of variations with fractional derivatives, *In I. Podlubny, B. M. Vinagre Jara, YQ. Chen, V. Feliu Batlle, I. Tejado Balsera (Eds.), Proceedings of FDA'10. The 4th IFAC Workshop Fractional Differentiation and its Applications. Badajoz, Spain (8-20 October 2010)*, Article no. FDA10-059, 6 pages, 2010.
10. Konjik, S., *Symmetries in Non-smooth Settings, Generalized Colombeau and Fractional Symmetries*, VDM Verlag Dr. Müller, Saarbrücken, 2009.
9. Atanacković, T. M., Konjik, S., Pilipović, S., Simić, S., Variational Problems with Fractional Derivatives: Invariance Conditions and Nöther's Theorem, *Nonlinear Anal., Theory Methods Appl.*, 71(5-6), 1504-1517, 2009.
8. Konjik, S., Kunzinger, M., Oberguggenberger, M., Foundations of the calculus of variations in generalized function algebras, *Acta Appl. Math.*, 103(2), 169-199, 2008.
7. Atanacković, T. M., Konjik, S., Pilipović, S., Variational Problems with Fractional Derivatives: Euler-Lagrange Equations, *J. Phys A: Math. Theor.*, 41, 095201, 2008.
6. Konjik, S., Kunzinger, M., Generalized group actions in a global setting, *J. Math. Anal. Appl.*, 322(1), 420-436, 2006.
5. Konjik, S., Kunzinger, M., Group invariants in algebras of generalized functions, *Integral Transforms Spec. Funct.*, 17(2-3) 77-84, 2006.
4. Konjik, S., Symmetries of conservation laws, *Publ. Inst. Math., Nouv. Sér.*, 77(91), 29-51, 2005.
3. Konjik, S., *Group Analysis and Variational Symmetries for Non-smooth Problems (PhD Thesis)*, Fakultät für Mathematik, Universitäat Wien, 2008.
2. Konjik, S., *Grupe simetrija sistema zakona održanja (master thesis)*, Univerzitet u Novom Sadu, Prirodno-matematički fakultet, 2003.
1. Konjik, S., Dedović, N., *Matematika – zbirka zadataka za studente Poljoprivrednog fakulteta*, Univerzitet u Novom Sadu, Poljoprivredni fakultet, 2007., drugo dopunjeno izdanje 2011.